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Combinatorics of the K-theory of affine Grassmannians

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Abstract

We introduce a family of tableaux that simultaneously generalizes the tableaux used to characterize Grothendieck polynomials and k-Schur functions. We prove that the polynomials drawn from these tableaux are the affine Grothendieck polynomials and k-K-Schur functions – Schubert representatives for the K-theory of affine Grassmannians and their dual in the nil Hecke ring. We prove a number of combinatorial properties including Pieri rules.

MSC: primary 05E05, 05E10; secondary 14N35, 17B65

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1. Introduction and background

Many problems in geometry and representation theory have been solved using the combinatorics behind Schur functions. Natural combinatorics associated to the more general families of Grothendieck polynomials and k-Schur functions has similarly led to an understanding of geometric and representation theoretic questions. Here we explore the underlying combinatorics of two families of affine Grothendieck polynomials.

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1.1. Schur functions

The Schur functions s_λ are homogeneous symmetric functions that form a fundamental basis for the symmetric function space Λ. Their study has profoundly influenced both practical and theoretical aspects of many fields. At the turn of the century, Schur functions were recognized to be the irreducible characters of the complex linear group. It was later revealed that combinatorics is deeply ingrained in their theory and can be applied to diverse problems in representation theory, geometry, physics, and beyond.

The combinatorics of Schur functions appears immediately with their very definition as the weight generating function of semi-standard tableaux:

$$s_\lambda = \sum_{\text{shape}(T)=\lambda} x^{\text{weight}(T)}. \tag{1}$$

They satisfy many beautiful combinatorial properties. For example, the *Pieri rule* for computing the Schur expansion

$$h_\ell s_\mu = \sum s_\lambda \tag{2}$$

is a simple matter of adding horizontal ℓ-strips to μ. More generally, the *Littlewood–Richardson coefficients*, occurring in the expansion

$$s_\nu s_\mu = \sum_\lambda c_{\nu\mu}^\lambda s_\lambda, \tag{3}$$

can be computed by counting Yamanouchi skew tableaux. Schur functions, indexed by partitions, also have deep ties to the theory of partitions as can be seen when working with the algebra automorphism defined by $\omega e_\ell = h_\ell$. This involution acts simply on a Schur function by

$$\omega s_\lambda = s_{\lambda'}, \tag{4}$$

where λ' is the transpose of shape λ.

Combinatorial Schur theory has applications in many fields. An illustrative example is given by the geometric problem of calculating intersection numbers on the Grassmannian variety $Gr_{\ell n}$ of ℓ-dimensional subspaces in $\mathbb{C}^n$. The cohomology ring $H^*(Gr_{\ell n})$ has a basis of Schubert classes σ_λ, indexed by partitions that fit inside an $\ell \times (n-\ell)$ rectangle. Intersection numbers of Schubert varieties are given by the structure constants $c_{\lambda\mu}^\nu$ in this basis;

$$\sigma_\lambda \cup \sigma_\mu = \sum_{\nu \subseteq \ell\times(n-\ell)} c_{\lambda\mu}^\nu \sigma_\nu. \tag{5}$$

Remarkably, an explicit understanding of the cohomology ring and of these intersection numbers is gained through the combinatorics of Schur functions. In particular, there is an isomorphism from a quotient of Λ to $H^*(Gr_{\ell n})$ where s_λ maps to the Schubert class σ_λ if λ fits in $\ell \times (n-\ell)$ and to zero otherwise. The structure constants of $H^*(Gr_{\ell n})$ are thus none other than the Littlewood–Richardson coefficients (3) for Schur function products. The beauty of this identification is that the two theories can be studied in parallel, where combinatorics such as the Pieri and Littlewood–Richardson rules provide elegant solutions to problems in both areas.

1.2. Grothendieck polynomials

Lascoux and Schützenberger introduced the Grothendieck polynomials in [23] as representatives for the K-theory classes determined by structure sheaves of Schubert varieties. Grothendieck polynomials are connected to combinatorics, representation theory, and algebraic geometry in a way that mimics ties between these theories and Schur functions (e.g. [4,8,20,6]). This study leads to a generalization of Schubert calculus where combinatorics is again at the forefront. For example, the stable Grothendieck polynomials G_λ are inhomogeneous symmetric polynomials whose lowest homogeneous component is a Schur function. They are characterized by Buch [3] as the weight generating function of set-valued tableaux:

$$G_\lambda = \sum_{\substack{T \text{ set-valued} \\ \mathrm{shape}(T)=\lambda}} (-1)^{|\lambda|-|\mathrm{weight}(T)|} x^{\mathrm{weight}(T)}. \tag{6}$$

The Pieri rules are in terms of binomial numbers [24] and there is a natural generalization for Yamanouchi tableaux [3] that gives a combinatorial rule for the structure constants.

Contrary to Schur functions, Grothendieck polynomials are not self-dual with respect to the Hall-inner product $\langle\,,\rangle$. This gives rise to the family of polynomials g_λ, dual to G_μ, whose *top* homogeneous component is a Schur function. Although less well-explored, the theory of these *dual Grothendieck polynomials* is equally as interesting.

1.3. k-Schur functions

There is a refinement of Schur functions along other lines that arose circuitously in a study of Macdonald polynomials [15]. Pursuant work led to the basis of *k-Schur functions* for $\mathbb{Z}[h_1,\ldots,h_k]$ that satisfies properties analogous to (1)–(5) in an affine setting. These functions, $s_\lambda^{(k)}$, are homogeneous symmetric functions indexed only by partitions λ where $\lambda_1 \leqslant k$.

Geometrically, it was proven that the k-Schur functions $s_\lambda^{(k)}$ reveal the structure of the quantum cohomology of the Grassmannian and the homology of affine Grassmannians analogous to the Schur role in $H^*(Gr_{\ell n})$. Quantum cohomology originated in string theory and symplectic geometry as a means to study Gromov–Witten invariants. It was shown in [18] that k-Schur structure constants are certain Gromov–Witten invariants and thus calculation in the quantum cohomology of the Grassmannian can be reduced to computing the product of k-Schur functions. It was then more generally shown in [10] that k-Schur functions are the Schubert basis for homology of the affine Grassmannian. Again, combinatorics behind k-Schur functions is key to their study, as well as to the geometry. Properties such as their Pieri rule are proven in [17].

A second affine analog for Schur functions was introduced in [18]. These *dual k-Schur functions* (or *affine Schur functions*) can be defined by $\langle s_\lambda^{(k)}, \mathfrak{S}_\mu^{(k)}\rangle = \delta_{\lambda\mu}$. These have significance in geometry as the Schubert basis for the cohomology of affine Grassmannians and are also studied by way of combinatorial identities. For example, dual k-Schur functions are the weight generating functions of tableaux introduced in [16] that are in bijection with elements of the type-A affine Weyl group,

$$\mathfrak{S}_\lambda^{(k)} = \sum_{\substack{T \text{ k-tableaux} \\ \mathrm{shape}(T)=\mathfrak{c}(\lambda)}} x^{\mathrm{weight}(T)}, \tag{7}$$

and they satisfy a Pieri rule (established in [11]).

1.4. Affine Grothendieck polynomials

The extension of ideas in k-Schur theory to an inhomogeneous setting underlies our investigation of affine combinatorics in the K-theoretic framework. We present a family of *affine set-valued tableaux* (or *affine s-v tableaux*) that simultaneously generalizes those used to characterize Grothendieck polynomials *and* k-Schur functions. We produce a bijection between affine s-v tableaux and certain elements that arise from the affine nil Hecke algebra. From this, we prove that the polynomials drawn from these tableaux:

$$G_\lambda^{(k)} = \sum_{\substack{T \text{ affine s-v tableaux} \\ \text{shape}(T)=\mathfrak{c}(\lambda)}} (-1)^{|\lambda|+|\text{weight}(T)|} x^{\text{weight}(T)}, \tag{8}$$

are affine stable Grothendieck polynomials introduced in [9]. We also study their dual with respect to the Hall-inner product, the k-K-Schur functions $g_\lambda^{(k)}$.

We prove that the affine s-v tableaux associated to integer $k > 0$ contain k-tableaux as a subset and reduce to set-valued tableaux when k is large. As a consequence, affine Grothendieck polynomials and k-K-Schur functions reduce to Grothendieck polynomials and their dual in a limiting case. Moreover, the term of lowest degree in $G_\lambda^{(k)}$ is the dual k-Schur function $\mathfrak{S}_\lambda^{(k)}$ and the highest term of $g_\lambda^{(k)}$ is the k-Schur function $s_\lambda^{(k)}$.

We also give a number of combinatorial properties for the k-K-Schur functions such as Pieri rules. In particular, for k-bounded partition λ and $r \leqslant k$,

$$g_r^{(k)} g_\lambda^{(k)} = \sum_{(\mu,\rho)\in\mathcal{H}_{\lambda,r}^k} (-1)^{r+|\lambda|-|\mu|} g_\mu^{(k)} \quad \text{and} \tag{9}$$

$$g_{1^r}^{(k)} g_\lambda^{(k)} = \sum_{(\mu,\rho)\in\mathcal{E}_{\lambda,r}^k} (-1)^{r+|\lambda|-|\mu|} g_\mu^{(k)}, \tag{10}$$

where the elements of $\mathcal{H}_{\lambda,r}^k$ and $\mathcal{E}_{\lambda,r}^k$ are obtained by way of an affine set-valued notion of horizontal and vertical strips, respectively. In addition to extending (1) and (2) to the affine K-theoretic setting, we find that k-K-Schur functions satisfy a natural analog to (4). The image of $g_\lambda^{(k)}$ under an involution Ω on Λ is simply another k-K-Schur function:

$$\Omega g_\lambda^{(k)} = g_{\lambda^{\omega_k}}^{(k)}, \tag{11}$$

where λ^{ω_k} is a certain unique "k-conjugate" partition associated to λ.

Our results establish that the k-K-Schur functions and affine Grothendieck polynomials are the affine K-theoretic Schur functions in a combinatorial sense. Lam, Schilling, and Shimozono show in [14] that these polynomials satisfy an analog geometrically along the lines of (5) and (3). We thus have a combinatorial framework within which to study this geometry as has been so fruitful in classical Schur function theory.

1.5. Related work

This study grew out of an FRG problem solving session in Viña del Mar, Chile (2008) during which Thomas Lam posed the problem of exploring polynomials to play the Schur role in an affine K-theoretic setting. Our results, establishing that the affine Grothendieck polynomials and k-K-Schur functions are the appropriate candidate, are obtained purely combinatorially. Recent work of Lam, Schilling, and Shimozono [14] carries out a similar investigation from the geometric viewpoint and they prove that $G_\lambda^{(k)}$ and $g_\lambda^{(k)}$ are Schubert representatives for K-theory classes of the affine Grassmannian and their dual in the nil Hecke ring, respectively.

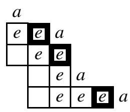
Another direction explores ties between the theories of Grothendieck and Macdonald polynomials. k-Schur functions are conjectured to be the $t=1$ specialization of atoms, a family of polynomials that arose in the study of Macdonald polynomials [15]. In joint work with Jason Bandlow [1], we prove that the Macdonald polynomials can be expanded positively (up to degree-alternating sign) in terms of the $\{g_\lambda\}_\lambda$ and in terms of $\{G_\lambda\}_\lambda$ basis. We introduce statistics on set-valued tableaux and on reverse plane partitions that naturally generalizes the Lascoux–Schützenberger charge [22] and show the statistics characterize the g and G-expansions of Hall–Littlewood polynomials.

2. Definitions

2.1. Partitions

A partition is an integer sequence $\lambda = (\lambda_1 \geqslant \cdots \geqslant \lambda_m > 0)$ whose degree is $|\lambda| = \lambda_1 + \cdots + \lambda_m$ and whose length $\ell(\lambda)$ is m. The (Ferrers) shape of the partition λ is a left- and bottom-justified array of 1×1 square cells in the first quadrant of the coordinate plane, with λ_i cells in the ith row from the bottom. The coordinates (i, j) are given to the cell in the ith row and jth column. We say that $\lambda \subseteq \mu$ when $\lambda_i \leqslant \mu_i$ for all i. When $\rho \subseteq \gamma$, the skew shape γ/ρ is the shape consisting of cells $\{(i,j)\colon\ \rho_i < j \leqslant \gamma_i\}$. The conjugate λ' of partition λ is the shape obtained by reflecting λ about the diagonal. Dominance order $\lambda \trianglerighteq \mu$ on partitions is defined by $|\lambda| = |\mu|$ and $\lambda_1 + \cdots + \lambda_i \geqslant \mu_1 + \cdots + \mu_i$ for all i.

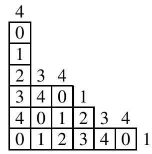
Given a partition γ, a *γ-removable* corner is a cell $(i,j) \in \gamma$ with $(i, j+1), (i+1, j) \notin \gamma$ and a *γ-addable* corner is a square $(i,j) \notin \gamma$ with $(i, j-1), (i-1, j) \in \gamma$. We should note that $(1, \gamma_1 + 1), (\ell(\gamma)+1, 1)$ are addable corners. A cell $(i,j) \in \gamma$ where $(i+1, j+1) \notin \gamma$ is called *extremal*. In particular, removable corners are extremal. In the skew shape $(5,3,3,2)/(1,1)$ below, all addable corners are labeled by a, extremals labeled by e, and removable corners are framed.



(12)

Given a cell $c = (i,j) \in \gamma$, the cell $(i, j+1)$ is said to be right-adj to c and $(i, j-1)$ is left-adj to c.

The hook-length of a cell c in a partition γ is the number of cells above and to the right of c, including c itself. A *p-core* is a partition that does not contain any cells with hook-length p and we let $\mathcal{C}^p$ be the collection of p-cores. The *p-residue* of square (i,j) is $j - i \mod p$. For example, labeling the 5-core $(6,4,3,1,1,1)$ and some nearby squares with their 5-residues gives



Hereafter we work with a fixed integer $k > 0$ and all cores/residues are $k+1$-cores/$k+1$-residues. For convenience, we refer to a corner of residue i as an *i-corner*. For a skew shape D, the set of residues of the cells in D is denoted by $Res(D)$ and when a cell c of D has residue i, we say $c \in D_{\downarrow i}$. Similarly, to indicate when a letter x in tableau T lies in a cell of residue i, we say $x \in T_{\downarrow i}$.

Several basic properties of cores will be used throughout. For example, Property 15 of [16] is particularly useful in our study:

Property 1. For a core γ and fixed $0 \leqslant i \leqslant k$, any cell $c \in \gamma_{\downarrow i}$ in row r,

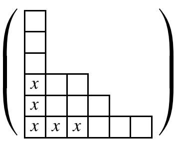
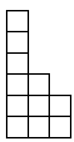
- if c lies at the end of its row then all extremals of $\gamma_{\downarrow i}$ in a row higher than r lie at the end of their row,
- if c lies at the top of its column then all extremals of $\gamma_{\downarrow i}$ in a row lower than r lie at the top of their column.

Note then that a core never has both an addable and a removable corner of the same residue. Further, given a core γ and any addable i-corner, the shape obtained by adding *all* i-corners to γ is also core.

A partition λ is *k-bounded* if $\lambda_1 \leqslant k$. We denote the set of all such partitions by $\mathcal{P}^k$. Notions for a composition α are similarly defined; the length $\ell(\alpha)$ is the number of parts in α and α is *k-bounded* if $\alpha_i \leqslant k$. A bijection $\mathfrak{p}$ from $k+1$-cores to k-bounded partitions was defined in [16] by the map

$$\mathfrak{p}(\gamma) = (\lambda_1, \lambda_2, \ldots, \lambda_\ell),$$

where λ_i is the number of cells in row i of γ that have a k-bounded hook-length (i.e. not exceeding k). For example, if $k = 5$,

$\mathfrak{p}$  $=$  (13)

where cells of γ whose hook-length exceeds 5 are marked by x. We denote the inverse by $\mathfrak{c} = \mathfrak{p}^{-1}$. Note that $|\lambda|$ is the number of cells in $\mathfrak{c}(\lambda)$ with k-bounded hook-length. This bijection induces a natural refinement of conjugation operator on partitions. In particular, the *k-conjugation* of k-bounded partitions $\omega^k : \mathcal{P}^k \to \mathcal{P}^k$ was defined in [16] by

$$\omega^k : \lambda \to \mathfrak{p}\big(\mathfrak{c}(\lambda)'\big). \tag{14}$$

Remark 2. (See Proposition 22 in [16].) Given a core β, let i denote the residue of any β-addable corner. If γ is obtained by adding all i-corners to β then $|\mathfrak{p}(\gamma)| = |\mathfrak{p}(\beta)| + 1$.

2.2. Symmetric functions and tableaux

Let Λ denote the ring of symmetric functions, generated by the elementary symmetric functions $e_r = \sum_{i_1<\cdots<i_r} x_{i_1}\cdots x_{i_r}$, or equivalently by the complete functions

$$h_r = \sum_{i_1\leqslant\cdots\leqslant i_r} x_{i_1}\cdots x_{i_r}.$$

The *Hall-inner product* on Λ is defined by

$$\langle h_\lambda, m_\mu\rangle = \delta_{\lambda\mu},$$

where m_μ is a monomial symmetric function. Complete details on symmetric functions can be found in e.g. [25,27,21].

Schur functions s_λ are the orthonormal basis with respect to the Hall-inner product. They can be combinatorial defined using (semi-standard) *tableaux*. The conventional definition of a tableau is the filling of each cell in a Ferrers shape with integers so that numbers strictly increase in columns and do not decrease in rows. The subset of *standard* tableaux are those where integers strictly increase in both columns *and* rows. The weight of tableau T is the composition $\mathrm{weight}(T)=\alpha$ where α_i is the number of cells of T containing an i. Schur functions are the weight generating functions of tableaux:

$$s_\lambda = \sum_{\mathrm{shape}(T)=\lambda} x^{\mathrm{weight}(T)}, \tag{15}$$

where $x^\alpha = x_1^{\alpha_1}x_2^{\alpha_2}\cdots$, for a composition α.

We will sometimes equivalently view a tableau as a sequence of shapes differing by horizontal strips where a *horizontal r-strip* is a skew shape with r cells, each lying in its own column. To be precise, a tableau of weight α and shape λ is a sequence

$$\emptyset = \lambda^{(0)} \subseteq \lambda^{(1)} \subseteq \lambda^{(2)} \subseteq \cdots = \lambda^{(0)} \subseteq \lambda^{(\ell(\alpha))} = \lambda,$$

where $\lambda^{(x)}/\lambda^{(x-1)}$ is a horizontal α_x-strip, for $x=1,\ldots,\ell(\alpha)$.

2.3. Set-valued tableaux

The weight generating function for stable *Grothendieck polynomials* is given in [3] using *set-valued tableaux*. In analogy to the conventional definition of tableaux, a set-valued tableau is a filling of each cell in a Ferrers shape with a set of integers, where a set X below (west of) Y satisfies $\max X < (\leqslant) \min Y$. A set-valued tableau is *standard* if it has the stronger condition that $\max X < \min Y$ for X below or west of Y. The weight of a set-valued tableau T is the composition α where α_i is the number of cells in T containing an i. For any partition λ, the symmetric Grothendieck polynomial is

$$G_\lambda = \sum_{\substack{\textit{set valued } T\\ \mathrm{shape}(T)=\lambda}} (-1)^{|\mathrm{weight}(T)|+|\lambda|} x^{\mathrm{weight}(T)}. \tag{16}$$

A set-valued tableau where every cell contains a set of cardinality one is a semi-standard tableau of the same shape λ and weight α. This occurs iff $|\alpha| = |\lambda|$ and otherwise $|\alpha| > |\lambda|$. If $\mathcal{K}_{\lambda\alpha}$ enumerates the set-valued tableaux of shape λ and weight α, then

$$G_\lambda = \sum_{|\mu| \geqslant |\lambda|} (-1)^{|\lambda|+|\mu|} \mathcal{K}_{\lambda\mu} m_\mu = s_\lambda + \text{terms of higher degree}. \tag{17}$$

Dual Grothendieck polynomials g_λ (e.g. [24,26,13]) can be defined by

$$\langle g_\lambda, G_\mu \rangle = \delta_{\lambda\mu}.$$

Duality and (17) imply that

$$h_\mu = \sum_{|\lambda| \leqslant |\mu|} (-1)^{|\lambda|+|\mu|} \mathcal{K}_{\lambda\mu} g_\lambda. \tag{18}$$

In fact, since the transition matrix $\|\mathcal{K}_{\lambda\mu}\|_{\lambda\mu}$ is unitriangular, the system obtained from this expression over all partitions can be inverted and used to characterize the $\{g_\lambda\}$. Inverting (18) also implies that

$$g_\lambda = s_\lambda + \text{terms of lower degree} \tag{19}$$

by the triangularity of h_μ in terms of Schur functions.

2.4. *k-tableaux*

Let γ be a $k+1$-core and let $\alpha = (\alpha_1, \ldots, \alpha_r)$ be a composition of $|\mathfrak{p}(\gamma)|$. A "*k-tableau*" of shape γ and weight α is a semi-standard filling of γ with integers $1, 2, \ldots, r$ such that the collection of cells filled with letter x are labeled by exactly α_x distinct $k+1$-residues.

Example 3. The 3-tableaux of weight $(1,3,1,2,1,1)$ and shape $(8,5,2,1)$ are:

$$\begin{array}{llllllll} 5 \\ 4 & 6 \\ 2 & 3 & 4 & 4 & 6 \\ 1 & 2 & 2 & 2 & 3 & 4 & 4 & 6 \end{array} \qquad \begin{array}{llllllll} 6 \\ 4 & 5 \\ 2 & 3 & 4 & 4 & 5 \\ 1 & 2 & 2 & 2 & 3 & 4 & 4 & 5 \end{array} \qquad \begin{array}{llllllll} 4 \\ 3 & 6 \\ 2 & 4 & 4 & 5 & 6 \\ 1 & 2 & 2 & 2 & 4 & 4 & 5 & 6 \end{array} \tag{20}$$

Remark 4. When $k \geqslant h(\gamma)$, a k-tableau T of shape γ and weight α is a semi-standard tableau of weight α since no two diagonals of T can have the same residue.

The symmetric family of *dual k-Schur functions* was introduced in [18] and defined to be the weight generating function of k-tableaux: for any k-bounded partition λ,

$$\mathfrak{S}_\lambda^{(k)} = \sum_{\substack{T \text{ } k\text{-tab} \\ \text{shape}(T) = \mathfrak{c}(\lambda)}} x^{\text{weight}(T)}. \tag{21}$$

It is shown in [16] that the number $K^{(k)}_{\mu\alpha}$ of k-tableaux of shape $\mathfrak{c}(\mu)$ and weight α satisfies the property

$$K^{(k)}_{\mu\lambda} = 0 \quad \text{when } \mu \not\trianglerighteq \lambda \quad \text{and} \quad K^{(k)}_{\mu\mu} = 1, \tag{22}$$

for any $\lambda, \mu \in \mathcal{P}^k$. Therefore, the monomial expansion has the form

$$\mathfrak{S}^{(k)}_{\lambda} = m_\lambda + \sum_{\substack{\mu \in \mathcal{P}^k \\ \mu \lhd \lambda}} K^{(k)}_{\lambda\mu} m_\mu, \tag{23}$$

revealing that $\{\mathfrak{S}^{(k)}_{\lambda}\}_{\lambda \in \mathcal{P}^k}$ forms a basis for

$$\Lambda/\mathcal{I}^k \quad \text{where } \mathcal{I}^k = \langle m_\lambda \colon \lambda_1 > k \rangle.$$

This space is naturally paired with

$$\Lambda^{(k)} = \mathbb{Z}[h_1, h_2, \ldots, h_k].$$

Since $\langle h_i \colon i > k \rangle$ is dual to $\mathcal{I}^k$ with respect to the Hall-inner product, $\Lambda^{(k)}$ is dual to $\Lambda/\mathcal{I}^k$. The basis for $\Lambda^{(k)}$ that is dual to $\{\mathfrak{S}^{(k)}_{\lambda}\}_{\lambda \in \mathcal{P}^k}$ is made up of the *k-Schur functions* $s^{(k)}_\lambda$. Since the matrix $\| K^{(k)}_{\lambda\mu} \|_{\lambda,\mu \in \mathcal{P}^k}$ is invertible, the system

$$h_\lambda = s^{(k)}_\lambda + \sum_{\mu \colon \mu \rhd \lambda} K^{(k)}_{\mu\lambda} s^{(k)}_\mu \quad \text{for all } \lambda_1 \leqslant k \tag{24}$$

can be taken as the definition of k-Schur functions [17].

3. Affine set-valued tableaux

3.1. Definition

In this section, we introduce and derive properties for a family of tableaux that generalizes both k-tableaux and set-valued tableaux. In subsequent sections, from these tableaux we will extract an inhomogeneous generalization of (dual) k-Schur functions *and* an affine analog of (dual) Grothendieck polynomials.

Let $T_{\leqslant x}$ denote the subtableau obtained by deleting all letters larger than x from T. For example,

$$T = \begin{array}{|c|c|c|c|} \cline{1-1} \{7\} \\ \cline{1-2} \{2,5\} & \{6\} \\ \hline \{1\} & \{2,3\} & \{4\} & \{4,6\} \\ \hline \end{array} \qquad T_{\leqslant 4} = \begin{array}{|c|c|c|c|} \cline{1-1} \{2\} \\ \hline \{1\} & \{2,3\} & \{4\} & \{4\} \\ \hline \end{array}$$

Definition 5. A standard affine set-valued tableau T of degree n is a set-valued filling such that, for each $1 \leqslant x \leqslant n$, shape$(T_{\leqslant x})$ is a core and the cells containing an x form the set of removable i-corners of $T_{\leqslant x}$, for some residue i.

Example 6. With $k = 2$, the standard affine s-v tableaux of degree 5 with shape $\mathfrak{c}(2,1,1) = (3,1,1)$ are

$$
\begin{array}{|c|} \hline \{3,4\}_1 \\ \hline \{2\}_2 \\ \hline \end{array}\!\!
\begin{array}{l} \\ \\ \end{array}
\quad \cdots
$$

$$
\begin{array}{lll} \{3,4\}_1 & & \\ \{2\}_2 & & \\ \{1\}_0 & \{3,4\}_1 & \{5\}_2 \end{array}
\quad
\begin{array}{lll} \{3,5\}_1 & & \\ \{2\}_2 & & \\ \{1\}_0 & \{3\}_1 & \{4\}_2 \end{array}
\quad
\begin{array}{lll} \{5\}_1 & & \\ \{4\}_2 & & \\ \{1,2\}_0 & \{3\}_1 & \{4\}_2 \end{array}
\quad
\begin{array}{lll} \{4\}_1 & & \\ \{3\}_2 & & \\ \{1,2\}_0 & \{4\}_1 & \{5\}_2 \end{array}
\quad
\begin{array}{lll} \{4\}_1 & & \\ \{2,3\}_2 & & \\ \{1\}_0 & \{4\}_1 & \{5\}_2 \end{array}
\tag{25}
$$

$$
\begin{array}{lll} \{5\}_1 & & \\ \{4\}_2 & & \\ \{1\}_0 & \{2,3\}_1 & \{4\}_2 \end{array}
\quad
\begin{array}{lll} \{5\}_1 & & \\ \{3,4\}_2 & & \\ \{1\}_0 & \{2\}_1 & \{3,4\}_2 \end{array}
\quad
\begin{array}{lll} \{4\}_1 & & \\ \{3\}_2 & & \\ \{1\}_0 & \{2\}_1 & \{3,5\}_2 \end{array}
\tag{26}
$$

Here, the subscript on a set is the residue of the cell containing that set.

An alternative to the conventional definition of tableaux is to instead define a semi-standard tableau to be a standard tableau with certain conditions on its *reading word*. The reading word of a tableau is obtained by taking the entries of T from top to bottom and left to right. A tableau of weight α is then a standard tableau having increasing reading words in the alphabets

$$\mathcal{A}_{\alpha,x} = \left[1 + \Sigma^{x-1}\alpha,\, \Sigma^{x}\alpha\right] \quad \text{where } \Sigma^{x}\alpha = \sum_{i \leqslant x} \alpha_i, \tag{27}$$

for $x = 1, \ldots, \ell(\alpha)$.

This arises through α-*standardization* of a semi-standard tableau T. The process constructs a standard tableau from T as follows: the α_1 cells that contain 1 in T are replaced by the numbers $1, 2, \ldots, \alpha_1$, increasingly from left to right. Then the cells that originally contained 2's are replaced by $\alpha_1 + 1, \ldots, \alpha_1 + \alpha_2$, increasingly from left to right, and so on.

We can define set-valued tableaux from this viewpoint as well by applying the same α-standardization process to set-valued tableaux. In this convention, a set-valued tableau T of weight α is a standard set-valued tableau with increasing reading words in the alphabets (27), where the reading word is obtained by reading letters from a cell in decreasing order (and as usual, cells are taken from top to bottom and left to right). The characterization of set-valued tableaux described here will be referred to as the "*standardized*" convention.

It is in this spirit that we have defined our affine K-theoretic generalization of tableaux. However, since letters in standard affine s-v tableaux can occur with multiplicity, we must consider the *lowest reading word* in $\mathcal{A}$, obtained by reading the lowest occurrence of the letters in $\mathcal{A}$ from top to bottom and left to right. Again, letters in the same cell are read in decreasing order. In Example 6, the lowest reading words in $\{1, \ldots, 5\}$ are 21435, 52134, 52134, 32145, 32145, 51324, 51243, 41253.

Definition 7. For any k-bounded composition α, an affine s-v tableau of weight α is a standard affine s-v tableau of degree $|\alpha|$ where, for each $1 \leqslant x \leqslant \ell(\alpha)$,

(1) the lowest reading word in $\mathcal{A}_{\alpha,x}$ is increasing,
(2) the letters of $\mathcal{A}_{\alpha,x}$ occupy α_x distinct residues,
(3) the letters of $\mathcal{A}_{\alpha,x}$ form a horizontal strip.

We let $\mathcal{T}^k(\lambda)$ denote the affine s-v tableaux of shape $\mathfrak{c}(\lambda)$ and let $\mathcal{T}_\alpha^k(\lambda)$ be the subset of those with weight α.

Note that the definition extends naturally to a family of skew affine s-v tableaux by deleting letters $1, \ldots, a$ from an affine s-v tableau and replacing each remaining letter x by $x - a$.

Example 8. The affine s-v tableaux in (26) of Example 6 all have weight $(2, 1, 1, 1)$ and comprise the set $\mathcal{T}^2_{(2,1,1,1)}(2, 1, 1)$.

3.2. Retrieving set-valued and k-tableaux

To justify that affine s-v tableaux are in fact an affine K-theoretic version of tableaux, we connect them to k-tableaux and set-valued tableaux.

Proposition 9. *For any k-bounded partition λ where $h(\lambda) \leqslant k$, the affine s-v tableaux of shape λ are simply the set-valued tableaux of shape λ (characterized in standardized convention).*

Proof. Given $h(\lambda) \leqslant k$, $\mathfrak{c}(\lambda) = \lambda$ and no two diagonals of λ have the same residue. Thus the lowest reading word of an affine s-v tableau T of weight α is simply its usual reading word and Condition (1) implies that T meets the increasing condition of a set-valued tableau in standardized convention. On the other hand, the α-standardized convention for a set-valued tableau of weight α ensures that the reading word of the α_x letters in $\mathcal{A}_{\alpha,x}$ is increasing and forms a horizontal strip. $\square$

To make the connection with k-tableaux, we need several basic properties of affine s-v tableaux. We say that x is *lonely* when a letter x occurs in cell without another letter.

Property 10. Given an affine s-v tableau T, $\mathrm{shape}(T_{\leqslant x})/\mathrm{shape}(T_{\leqslant x-1}) \neq \emptyset$ if and only if every x is lonely in $T_{\leqslant x}$, for any letter $x \in T$.

Proof. If $\mathrm{shape}(T_{\leqslant x}) = \mathrm{shape}(T_{\leqslant x-1})$ then no x can be lonely since $T_{\leqslant x-1}$ is obtained by deleting the letter x from $T_{\leqslant x}$. On the other hand, we will show that if $T_{\leqslant x}$ has some x that is not lonely then no x are lonely implying the shapes must be equal. Suppose $T_{\leqslant x}$ has a cell of some residue i containing x by itself and one containing x with another letter. Then $\mathrm{shape}(T_{\leqslant x-1})$ has both a removable and an addable i-corner. This contradicts that a core never contains an addable and removable i-corner. $\square$

Property 11. Given an affine s-v tableau T, if $\gamma^{(x)} = \mathrm{shape}(T_{\leqslant x})$ then

$$\left|\mathfrak{p}\left(\gamma^{(x)}\right)\right| = \left|\mathfrak{p}\left(\gamma^{(x-1)}\right)\right| + 1 \quad \text{for any } \gamma^{(x)} \neq \gamma^{(x-1)}.$$

Proof. Given T is an affine s-v tableau, there is an x in all removable i-corners of the core $\gamma^{(x)}$. If $\gamma^{(x)} \neq \gamma^{(x-1)}$ then these x are all lonely by Property 10. Therefore, $\gamma^{(x)}$ is $\gamma^{(x-1)}$ plus addable i-corners and the result follows from Remark 2. $\square$

From Property 11, an affine s-v tableau T of weight α has the property that $|\mathfrak{p}(\mathrm{shape}(T_{\leqslant x}))|$ is either $|\mathfrak{p}(\mathrm{shape}(T_{\leqslant x-1}))|$ or $|\mathfrak{p}(\mathrm{shape}(T_{\leqslant x-1}))| + 1$, for all $x = 1, 2, \ldots, |\alpha|$. Consequently:

Corollary 12. *If there is an affine s-v tableau of weight α and shape $\mathfrak{c}(\lambda)$, then $|\alpha| \geqslant |\lambda|$.*

We are now prepared to prove that the family of affine s-v tableaux includes k-tableaux.

Proposition 13. *The set of affine s-v tableaux $\mathcal{T}_\alpha^k(\lambda)$ when $|\alpha| = |\lambda|$ is the set of k-tableaux with weight α and shape $\mathfrak{c}(\lambda)$.*

Proof. Consider an affine s-v tableau T of weight α and shape $\gamma = \mathfrak{c}(\lambda)$ where $n = |\alpha| = |\lambda|$. For $x = 1, \ldots, n$, let $\lambda^{(x)} = \mathfrak{p}(\text{shape}(T_{\leqslant x}))$. Since $|\lambda^{(x)}| = |\lambda^{(x-1)}|$ or $|\lambda^{(x)}| = |\lambda^{(x-1)}| + 1$ by Property 11 and $|\lambda^{(n)}| = n$, $|\lambda^{(x)}| = |\lambda^{(x-1)}| + 1$ for all x. Thus, $\lambda^{(x-1)} \neq \lambda^{(x)}$ which implies that all x are lonely in $T_{\leqslant x}$ by Property 10, for all x. Therefore no cell of T has a set of cardinality more than one. The conditions on affine s-v tableau imply that replacing all letters in $\mathcal{A}_{\alpha,x}$ by x gives a semi-standard filling where x occupies α_x distinct residues and thus, T is a k-tableau.

On the other hand, given a k-tableau T of weight α, we can α-standardize T iteratively from $r = |\alpha|$ as follows: for the rightmost letter $x \in T$ that is not larger than r, let i denote its residue. Relabel every $x \in T$ that has residue i by r. Replace r by $r - 1$. It was shown in [16] that the resulting tableau U is a standard k-tableau and by construction, the lowest reading word in $\mathcal{A}_{\alpha,x}$ is clearly increasing. Therefore, since the letter x occupies α_x distinct residues and forms a horizontal strip in T, U meets the conditions of an affine s-v tableau of weight α. □

4. Alternate characterizations

4.1. Affine Weyl group characterization

In the theory of k-Schur functions, the discovery that k-tableaux are reduced words for Grassmannian permutations in the affine symmetric group was the spring board to understanding the k-Schur role in geometry. Here, we investigate a similar interpretation for affine s-v tableaux. To start, we consider the standard case and recall results in the k-tableaux case.

Let $\tilde{S}_{k+1}$ denote the affine Weyl group of A_k, generated by $\langle s_0, s_1, \ldots, s_k \rangle$ and satisfying the relations

$$\begin{aligned} s_i^2 &= 1 \quad \text{for all } i, \\ s_i s_{i+1} s_i &= s_{i+1} s_i s_{i+1} \quad \text{for all } i, \\ s_i s_j &= s_j s_i \quad \text{if } |i-j| > 1, \end{aligned} \tag{28}$$

where indices are taken modulo $k+1$ (hereafter, all indices are taken mod $k+1$). A word $i_1 i_2 \cdots i_m$ in the alphabet $\{0, 1, \ldots, k\}$ corresponds to the permutation $w \in \tilde{S}_{k+1}$ if $w = s_{i_1} \cdots s_{i_m}$. The length $\ell(w)$ of w is defined to be the length of its shortest word. Any word of this length is said to be *reduced* and we denote the set of all reduced words for w by $\mathcal{R}(w)$. The set $\tilde{S}_{k+1}^0$ of Grassmannian elements are the minimal length coset representatives of $\tilde{S}_{k+1}/S_{k+1}$, where S_{k+1} is the finite symmetric group. In fact, $w \in \tilde{S}_{k+1}^0$ iff every reduced word for w ends in 0.

Consider operators on set-valued tableaux defined for $i = 0, \ldots, k$ by

$$\mathfrak{s}_{i,x} : \hat{T} \to T,$$

where T is obtained by adding an x to all addable or all removable i-corners of $\hat{T}$. If there are no such corners, $\mathfrak{s}_{i,x}$ fixes $\hat{T}$. It turns out [16] that the set of reduced words for a fixed $w \in \tilde{S}_{k+1}^0$ is in bijection with the set of k-tableaux of some core shape γ. In particular, each reduced word $i_\ell i_{\ell-1} \cdots i_1$ for $w \in \tilde{S}_{k+1}^0$ is sent to the standard k-tableau $\mathfrak{s}_{i_\ell,\ell}\mathfrak{s}_{i_{\ell-1},\ell-1} \cdots \mathfrak{s}_{i_1,1}\emptyset$ on ℓ letters.

It is natural to work with operators defined on shapes, for $i = 0, \ldots, k$, by

$$\mathfrak{s}_i : \gamma \to \gamma + \text{its addable } i\text{-corners}.$$

These can be viewed as an affine analog of operators introduced in [3] and where the set-up follows that of [5].

Remark 14. A number of useful properties are satisfied by the $\mathfrak{s}_i$ operators.

(1) If β is a core with an addable i-corner, then $\mathfrak{s}_i(\beta)$ is a core and $|\mathfrak{p}(\mathfrak{s}_i(\beta))| = |\mathfrak{p}(\beta)| + 1$ by Remark 2.
(2) The bijection between $\tilde{S}_{k+1}^0$ and $\mathcal{C}^{k+1}$ defined by the map

$$\mathfrak{t} : w \to \gamma = \mathfrak{s}_{i_\ell} \cdots \mathfrak{s}_{i_1}\emptyset,$$

for any $i_\ell \cdots i_1 \in \mathcal{R}(w)$, has the property that $|\mathfrak{p}(\gamma)| = \ell$ since $i_\ell \cdots i_1$ corresponds to a k-tableaux of shape γ on ℓ letters.
(3) If $i_\ell i_{\ell-1} \cdots i_1$ is a reduced word for an affine Grassmannian permutation, then $\mathfrak{s}_{i_j}\mathfrak{s}_{i_{j-1}} \cdots \mathfrak{s}_{i_1}\emptyset$ has an addable i_{j+1}-corner for all $j < \ell$ by (1) and (2).
(4) When acting on cores, we have the relations

$$\mathfrak{s}_i^2 = \mathfrak{s}_i \quad \text{for all } i, \qquad \mathfrak{s}_i\mathfrak{s}_{i+1}\mathfrak{s}_i = \mathfrak{s}_{i+1}\mathfrak{s}_i\mathfrak{s}_{i+1}, \qquad \mathfrak{s}_i\mathfrak{s}_j = \mathfrak{s}_j\mathfrak{s}_i \quad \text{when } |i-j| > 1.$$

Denote the affine Grassmannian permutation associated to $\lambda \in \mathcal{P}^k$ by

$$w_\lambda = \mathfrak{t}^{-1}\big(\mathfrak{c}(\lambda)\big).$$

Remark 15. A quick way to construct w_λ from λ is to take the residues of λ read from *right* to *left* and top to bottom [16]. For example, $w_{(2,1,1)} \in \tilde{S}_3$ is

$$\begin{array}{|c|c} \hline 1 & \\ \cline{1-1} 2 & \\ \hline 0 & \multicolumn{1}{c|}{1} \\ \hline \end{array} \to s_1 s_2 s_1 s_0. \tag{29}$$

The relations in Remark 14(4) arise in the nil Hecke algebra. We will discuss this further in Section 5, but for now are interested in studying the equivalence classes of words under these relations. To be precise, we consider the set $\mathcal{W}(w_\lambda)$ of all words whose reduced expression is in $\mathcal{R}(w_\lambda)$ under the relations

$$\begin{aligned} u_i^2 &= u_i \quad \text{for all } i, \\ u_i u_{i+1} u_i &= u_{i+1} u_i u_{i+1} \quad \text{for all } i, \\ u_i u_j &= u_j u_i \quad \text{when } |i-j| > 1. \end{aligned} \tag{30}$$

Note that $\mathcal{R}(w_\lambda) = \{u \in \mathcal{W}(w_\lambda)\colon\ \ell(u) = \ell(w_\lambda)\}$.

Remark 16. For any $i_r \cdots i_1 \in \mathcal{W}(w_\lambda)$, $\mathfrak{s}_{i_r} \cdots \mathfrak{s}_{i_1}\emptyset$ has a removable or an addable i_m-corner for all $m \leqslant r$. To see this, let $\ell = \ell(w_\lambda)$ and iterate the following argument: if $i_{\ell+1} i_\ell \cdots i_1 \in \mathcal{W}(w_\lambda)$ then there is some t where $j_\ell \cdots j_t j_t \cdots j_1 \sim i_{\ell+1} i_\ell \cdots i_1$ (equivalent under the relations in (30)). Then $\mathfrak{s}_{j_\ell} \cdots \mathfrak{s}_{j_1}\emptyset$ has an addable j_m-corner by Remark 14(3) and thus $\mathfrak{s}_{j_\ell} \cdots \mathfrak{s}_{j_t}\mathfrak{s}_{j_t} \cdots \mathfrak{s}_{j_1}\emptyset$ has an addable or removable j_m-corner. Therefore, $\mathfrak{s}_{i_{\ell+1}} \cdots \mathfrak{s}_{i_1}\emptyset$ has an addable or a removable i_m-corner for any $m \leqslant \ell + 1$ by Remark 14(4).

Just as the k-tableaux represent reduced words for affine Grassmannian permutations, we find that standard affine s-v tableaux of fixed shape $\mathfrak{c}(\lambda)$ are none other than the words whose reduced expression is in $\mathcal{R}(w_\lambda)$.

Proposition 17. *For $\lambda \in \mathcal{P}^k$, there is a bijection*

$$\mathfrak{s} : \mathcal{W}(w_\lambda) \to \bigcup_{m \geqslant 1} \mathcal{T}^k_{1^m}(\lambda)$$

defined by $\mathfrak{s}(i_m i_{m-1} \cdots i_1) = \mathfrak{s}_{i_m,m}\mathfrak{s}_{i_{m-1},m-1} \cdots s_{i_1,1}\emptyset$.

Proof. Given $i_m \cdots i_1 \in \mathcal{W}(w_\lambda)$, we claim that $T = \mathfrak{s}(i_m \cdots i_1)$ is a standard affine s-v tableau of degree m. Since $T_{\leqslant 1} = \boxed{\{1\}}$, assume by induction that $T_{\leqslant m-1}$ is a standard affine s-v tableau of core shape γ. Remark 16 implies γ has either an addable or a removable i_m-corner and thus $T = T_{\leqslant m}$ is obtained by putting an m in all such corners. Therefore the shape of T is either γ or γ plus its addable i_m-corners; in both cases it is a core. Further, T has shape $\mathfrak{c}(\lambda)$ by Remark 14.

For any $T \in \mathcal{T}^k_{1^m}(\lambda)$, $T = \mathfrak{s}_{j_m,m} \cdots \mathfrak{s}_{j_1,1}\emptyset$ where j_a denotes the residue of letter a in T. Thus, $\mathfrak{c}(\lambda) = \mathfrak{s}_{j_m} \cdots \mathfrak{s}_{j_1}\emptyset$ and to prove $\mathfrak{s}$ is onto, we need $j_m \cdots j_1 \in \mathcal{W}(w_\lambda)$. Consider a reduced word $j'_\ell \cdots j'_1$ equivalent to $j_m \cdots j_1$. Remark 14(4) implies $\mathfrak{c}(\lambda) = \mathfrak{s}_{j'_\ell} \cdots \mathfrak{s}_{j'_1}\emptyset$ and therefore $j'_\ell \cdots j'_1 \in \mathcal{R}(w_\lambda)$.

To see that $\mathfrak{s}$ is 1–1, consider $\mathfrak{s}(i_m \cdots i_1) = \mathfrak{s}(\hat{i}_{\hat{m}} \cdots \hat{i}_1)$ for $i_m \cdots i_1, \hat{i}_{\hat{m}} \cdots \hat{i}_1 \in \mathcal{W}(w_\lambda)$. By definition, $\mathfrak{s}_{i,x}\hat{T}$ has an x in the addable or removable i-corners of $\hat{T}$. Therefore, if there is some (minimal) j where $i_j \neq \hat{i}_j$ clearly $\mathfrak{s}_{i_j,j}\hat{T} \neq \mathfrak{s}_{\hat{i}_j,j}\hat{T}$. $\square$

4.2. *Horizontal strip characterization*

In classical and k-tableaux theory, characterizing tableaux as a sequence of shapes satisfying horizontality conditions has many applications. Most notably, the Pieri rule for affine and usual Schur functions is readily apparent in this context. Moreover, given such an interpretation, the tie between the affine Weyl group and k-tableaux in the semi-standard case can be made. With this in mind, we set-out to coin affine s-v tableaux in terms of certain horizontal strips.

We start with the characterization of set-valued and k-tableaux as a sequence of shapes. For partitions $\beta \subseteq \gamma$ and ρ, a *set-valued* r-strip $(\gamma/\beta, \rho)$ is such that

(1) γ/ρ is a horizontal r-strip,
(2) β/ρ is a set of $r - |\gamma/\beta|$ β-removable corners.

The set-valued tableaux of weight α and shape λ are in bijection with the set of sequences having the following form:

$$(\emptyset, \emptyset) = (\lambda^{(0)}, \rho^{(0)}) \subset (\lambda^{(1)}, \rho^{(1)}) \subseteq (\lambda^{(2)}, \rho^{(2)}) \subseteq \cdots \subseteq (\lambda^{(\ell(\alpha))}, \rho^{(\ell(\alpha))}) = (\lambda, \rho^{(\ell(\alpha))})$$

where $(\lambda^{(x)}/\lambda^{(x-1)}, \rho^{(x)})$ is a set-valued α_x-strip, for $x = 1, \ldots, \ell(\alpha)$.

For $0 \leqslant r \leqslant k$, an *affine r-strip* γ/β is a horizontal strip where

(af1) γ and β are cores,
(af2) $|\mathfrak{p}(\gamma)| - |\mathfrak{p}(\beta)| = r$,
(af3) γ/β occupies r distinct residues.

It can be deduced from results in [16] that k-tableaux of weight α and shape γ are in bijection with chains of the form:

$$\emptyset = \gamma^{(0)} \subset \gamma^{(1)} \subseteq \gamma^{(2)} \subseteq \cdots \subseteq \gamma^{(\ell(\alpha))} = \gamma, \tag{31}$$

where $\gamma^{(x)}/\gamma^{(x-1)}$ is an affine α_x-strip, for $x = 1, \ldots, \ell(\alpha)$.

We impose additional conditions on these strips in order to view affine s-v tableaux in a similar way. Given cores $\beta \subseteq \gamma$, a cell of β that lies below a cell of γ is *γ-blocked*.

Definition 18. For $0 \leqslant r \leqslant k$, an "*affine set-valued*" r-strip $(\gamma/\beta, \rho)$ is such that

(asv1) γ/ρ is a horizontal strip,
(asv2) γ/β is an affine $r - m$-strip, where $m = |Res(\beta/\rho)|$,
(asv3) β/ρ is a subset of β-removable corners such that, for each $i \in Res(\beta/\rho)$, all β-removable i-corners that are not γ-blocked lie in β/ρ.

Remark 19. A priori, if $(\gamma/\beta, \rho)$ is an affine s-v r-strip then γ/β is an affine strip. In fact, γ/β is an affine r-strip when $\beta = \rho$. When $k = \infty$, an affine s-v strip is simply a set-valued strip since residues can occur at most once in a horizontal strip.

To prove that affine s-v tableaux can be characterized in terms of affine s-v strips, we need a number of properties about these and affine strips.

Proposition 20. *If γ/β is an affine r-strip and i is the residue of its rightmost cell, then $\hat{\gamma}/\beta$ is an affine $r - 1$-strip where $\hat{\gamma}$ is γ minus its removable i-corners.*

Proof. Since $|\mathfrak{p}(\hat{\gamma})| = |\mathfrak{p}(\gamma)| - 1$ by Remark 2, it suffices to prove that $\beta \subseteq \hat{\gamma}$. To this end, for i the residue of the rightmost cell in γ/β, we will prove that the γ-removable i-corners are exactly the cells of residue i in γ/β. Since γ/β is a horizontal strip whose rightmost cell has residue i then all cells of residue i in γ/β are γ-removable by Property 1 since they are all extremal in γ. It thus remains to show that any γ-removable i-corner is in γ/β.

Suppose by contradiction there is a γ-removable i-corner c that is also β-removable (choose the lowest). Note that it lies higher than any $a \in (\gamma/\beta)_{\downarrow i}$ since the cell that is beneath a has residue $i + 1$ and lies at the top of its column in β implying that all extremals of residue $i + 1$ lower than a are at the top of their column in β by Property 1. Then, the cell left-adj to any a must be in γ/β since otherwise it would end its row in β whereas the extremal left-adj to c does not (contradicting Property 1). Therefore, the column with c has more k-bounded hooks in β than in γ. Let $a_1, \ldots, a_r$ denote the lowest cells of residues $i_1, \ldots, i_r$, respectively, in γ/β. Each of these cells lies in a column with one more k-bounded hook in γ than in β by Property 1. In fact, these are the only columns where γ has more k-bounded hooks than β. Since $|\mathfrak{p}(\gamma)| - |\mathfrak{p}(\beta)| = r$ it must be that no column of β has more k-bounded hooks than in γ and we have our contradiction. □

Property 21. Let γ/β be an affine r-strip for some $r \leqslant k$. If β has a removable i-corner that is not γ-blocked then $i \notin Res(\gamma/\beta)$.

Proof. Consider a β-removable i-corner c, in some row r_c, that lies at the top of its column in γ. Suppose by contradiction that there is a $y \in (\gamma/\beta)_{\downarrow i}$ and let r_y be the lowest row containing such a cell. With $\gamma = \gamma^{(0)}$, let $\gamma^{(j)}$ be the core obtained by deleting all i_j-corners of $\gamma^{(j-1)}$, where i_j is the residue of the rightmost element in $\gamma^{(j-1)}/\beta$. Proposition 20 implies that $\gamma^{(j)}/\beta$ is an affine strip.

Note that $y \in (\gamma/\beta)_{\downarrow i}$ implies that there must be some t where $i_t = i$. Since y is the rightmost element of $\gamma^{(t-1)}/\beta$ and all cells in the same row and to the right of a β-removable cell are in γ/β, c lies at the end of its row in $\gamma^{(t-1)}$ (and thus in $\gamma^{(t)}$) if $r_c < r_y$. In this case, the cell below c is extremal in $\gamma^{(t)}$. However, the cell below y lies at the top of its column in $\gamma^{(t)}$ violating Property 1. Therefore, $r_c \geqslant r_y$. Note that the cell z left-adj to y lies at the end of row r_y in $\gamma^{(t)}$ and has residue $i-1$. Thus it is above an extremal cell of residue i whereas c is at the top of its column. Again by Property 1 we have a contradiction. □

Given an affine s-v strip $(\gamma/\beta, \rho)$, no cell in β/ρ is γ-blocked since γ/ρ is a horizontal strip. We therefore deduce that:

Corollary 22. *For any affine set-valued r-strip $(\gamma/\beta, \rho)$, $Res(\gamma/\beta) \cap Res(\beta/\rho) = \emptyset$.*

In particular, since $|Res(\gamma/\beta)| = r - m$ by the definition of affine strip, we have:

Corollary 23. *For any affine set-valued r-strip $(\gamma/\beta, \rho)$, $|Res(\gamma/\rho)| = r$.*

Property 24. If $(\gamma/\beta, \rho)$ is an affine s-v strip and i is the residue of the rightmost cell in γ/ρ, then $(\gamma/\rho)_{\downarrow i}$ is the set of γ-removable i-corners.

Proof. Let c be the rightmost cell of γ/ρ and let i denote its residue. If $c \in \gamma/\beta$, Property 20 implies that the set of γ-removable i-corners is $(\gamma/\beta)_{\downarrow i}$, which is exactly $(\gamma/\rho)_{\downarrow i}$ by Corollary 22. On the other hand, if $c \in \beta/\rho$, note that all cells in $(\gamma/\rho)_{\downarrow i}$ are γ-removable corners since the lowest cell of residue i in γ/ρ is at the end of its row implying all i-extremals are at the end of their row in γ by Property 1. If there is a γ-removable corner i-corner $\bar{c} \notin \gamma/\rho$, then $\bar{c} \in \rho \subseteq \beta \subseteq \gamma$ implies $\bar{c} \notin \beta/\rho$ and is a β-removable i-corner. Therefore $\bar{c}$ is γ-blocked by definition of affine s-v strip, contradicting that $\bar{c}$ is γ-removable. □

Proposition 25. *Consider the affine s-v strip $(\gamma/\beta, \rho)$ whose rightmost cell in γ/ρ has residue i. If $i \in Res(\gamma/\beta)$ then $(\hat{\gamma}/\beta, \rho)$ is an affine s-v strip where $\hat{\gamma}$ is γ minus its i-corners. Otherwise, $(\gamma/\beta, \hat{\rho})$ is an affine s-v strip where $\hat{\rho}$ is ρ plus its i-corners.*

Proof. Let c be the rightmost cell of γ/ρ and let i denote its residue. If $c \in \beta/\rho$, then $i \notin Res(\gamma/\beta)$ by Corollary 22 and there are no γ-addable i-corners since c is a γ-removable

i-corner by Property 24 and γ is a core. Therefore, all addable i-corners of ρ are in β/ρ implying $\hat{\rho} \subseteq \beta$ and thus that $(\gamma/\beta, \hat{\rho})$ is an affine s-v strip. On the other hand, if $c \in \gamma/\beta$ then Proposition 20 implies (asv1) and (asv2). Now suppose there are β-removables, $c_1 \in (\beta/\rho)_{\downarrow i}$ and $c_2 \notin (\beta/\rho)_{\downarrow i}$. Since $(\gamma/\beta, \rho)$ is an affine s-v strip, c_2 must be γ-blocked. If, by contradiction, c_2 is not $\hat{\gamma}$-blocked then $j = i + 1$. c lies at the end of its row in γ implying all higher i-extremals are at the end of their row in γ by Property 1. However, by horizontality of γ/ρ, the cell left-adj to c_1 is an i-extremal. $\square$

Now we are equipped to rephrase the definition of affine s-v tableaux. For $\lambda \in \mathcal{P}^k$, consider the set of pairs of shapes obtained by adding affine s-v strips to $\mathfrak{c}(\lambda)$,

$$\mathcal{H}^k_{\lambda,r} = \bigl\{(\mu, \rho)\colon \bigl(\mathfrak{c}(\mu)/\mathfrak{c}(\lambda), \rho\bigr) = \text{affine set-valued } r\text{-strip}\bigr\}.$$

Theorem 26. *For any $\lambda \in \mathcal{P}^k$ and k-bounded composition α, there is a bijection between $\mathcal{T}^k_\alpha(\lambda)$ and*

$$\bigl\{(\emptyset, \emptyset) \subset \bigl(\gamma^{(1)}, \rho^{(1)}\bigr) \subseteq \cdots \subseteq \bigl(\gamma^{(\ell(\alpha))}, \rho^{(\ell(\alpha))}\bigr)\colon \bigl(\mathfrak{p}\bigl(\gamma^{(x)}\bigr), \rho^{(x)}\bigr) \in \mathcal{H}^k_{\mathfrak{p}(\gamma^{(x-1)}),\alpha_x}\ \forall x \geqslant 1\bigr\},$$

where $\gamma^{(0)} = \emptyset$ and $\gamma^{(\ell(\alpha))} = \mathfrak{c}(\lambda)$.

Proof. ($\Leftarrow$): For $x = 1, \ldots, \ell(\alpha)$, we will construct T by filling the cells of $\gamma^{(x)}/\rho^{(x)}$ iteratively as follows: start with $N = \Sigma^x\alpha$. Let i denote the residue of the rightmost cell in $\gamma^{(x)}/\rho^{(x)}$ that contains no letter larger than N. Put letter N in all cells of $\gamma^{(x)}/\rho^{(x)}$ with residue i. Let $N = N - 1$ and repeat.

Since $\gamma^{(0)} = \emptyset$, $\gamma^{(1)}$ is the row shape (α_1) and thus $T_{\leqslant \alpha_1} = \boxed{1}\boxed{2}\boxed{\cdots}\boxed{\alpha_1}$ is an affine s-v tableau of weight (α_1). By induction, assume $T_{\leqslant \Sigma^{x-1}\alpha}$ is an affine s-v tableau of weight $(\alpha_1, \ldots, \alpha_{x-1})$. Note that Conditions (1) and (3) on affine s-v tableaux are met by construction since $\gamma^{(x)}/\rho^{(x)}$ is a horizontal strip. Further, Corollary 23 implies Condition (2). It thus remains to show that $T_{\leqslant \Sigma^x\alpha - j}$ is a set-valued tableau with $\Sigma^x\alpha - j$ in all removable corners of some residue and its shape $\tau^{(j)}$ is a core, for $j = 0, \ldots, \alpha_x - 1$.

Let $\beta = \gamma^{(x-1)}$ and $\eta^{(0)} = \rho^{(x)}$. Note that $\tau^{(0)} = \gamma^{(x)}$. By construction, there is an $N = \Sigma^x\alpha$ in all cells of $\tau^{(0)}/\eta^{(0)}$ with residue i_0, where i_0 is the residue of the rightmost cell c in $\tau^{(0)}/\eta^{(0)}$. Property 24 implies there is an N in exactly the $\tau^{(0)}$-removable i_0-corners.

Note that $\tau^{(1)}$ is $\tau^{(0)}$ minus cells containing a lonely N. If $c \in \tau^{(0)}/\beta$ this is $\tau^{(0)}$ minus its i_0-corners and otherwise $\tau^{(1)} = \tau^{(0)}$. In the later case, set $\eta^{(1)} = \eta^{(0)}$ plus its i_0-corners and otherwise let $\eta^{(1)} = \eta^{(0)}$. Proposition 25 thus implies that $(\tau^{(1)}/\beta, \eta^{(1)})$ is an affine s-v strip and in particular, $\tau^{(1)}$ is a core. By construction, there is an $N - 1$ in all cells of $\tau^{(1)}/\eta^{(1)}$ with residue i_1, where i_1 is the residue of the rightmost cell $c_1 \in \tau^{(1)}/\eta^{(1)}$. Thus, there is an $N - 1$ in exactly the $\tau^{(1)}$-removable i_1-corners by Property 24. Iterating this argument proves the claim.

($\Rightarrow$): Given $U \in \mathcal{T}^k_\alpha(\lambda)$, let $\gamma^{(x)} = \text{shape}(U_{\leqslant \Sigma^x\alpha})$ and $\rho^{(x)}$ be the shape obtained by deleting every *cell* containing an element of $\mathcal{A}_{\alpha,x}$ from $U_{\leqslant \Sigma^x\alpha}$, for $x = 1, \ldots, \ell(\alpha)$. We claim that each $(\gamma^{(x)}/\gamma^{(x-1)}, \rho^{(x)})$ is an affine set-valued α_x-strip.

Let $T = U_{\leqslant \Sigma^x\alpha}$ and note that

- $\gamma^{(x)}/\rho^{(x)}$ are the cells in T containing a letter in $\mathcal{A}_{\alpha,x}$,

- $\gamma^{(x-1)}/\rho^{(x)}$ are the cells in T containing a letter in $\mathcal{A}_{\alpha,x}$ *and* a letter weakly smaller than $\Sigma^{x-1}\alpha$,
- $\gamma^{(x)}/\gamma^{(x-1)}$ are the cells in T containing only letters larger than $\Sigma^{x-1}\alpha$.

The definition of affine s-v tableaux implies that $\gamma^{(x)}/\rho^{(x)}$ is a horizontal strip and that $\gamma^{(x)}$ and $\gamma^{(x-1)}$ are cores.

To show that $\gamma^{(x)}/\gamma^{(x-1)}$ is an affine $\alpha_x - |Res(\gamma^{(x-1)}/\rho^{(x)})|$-strip, we note it is a horizontal strip since $\rho^{(x)} \subseteq \gamma^{(x-1)}$. We claim that no letter lies in both $\gamma^{(x)}/\gamma^{(x-1)}$ and $\gamma^{(x-1)}/\rho^{(x)}$ implying (af3) since the α_x letters of $\mathcal{A}_{\alpha,x}$ each occupy a distinct residue and lie in $\gamma^{(x)}/\rho^{(x)}$. To this end, suppose $y = a + \Sigma^{x-1}\alpha \in \gamma^{(x)}/\gamma^{(x-1)}$. It must be lonely in $T_{\leqslant y}$ since $\gamma^{(x)}/\gamma^{(x-1)}$ has no letters smaller than $\Sigma^{x-1}\alpha$ and no two letters of $\mathcal{A}_{\alpha,x}$ share a cell. The proof of Property 10 then implies that *all* y are lonely in $T_{\leqslant y}$ and thus *all* $y \in \gamma^{(x)}/\gamma^{(x-1)}$. To verify (af2), let

$$\tau^{(a)} = \text{shape}(T_{\leqslant a+\Sigma^{x-1}\alpha}),$$

for $a = 1, \ldots, \alpha_x$. Since any $a + \Sigma^{x-1}\alpha \in \gamma^{(x)}/\gamma^{(x-1)}$ is lonely, $\tau^{(a)} \neq \tau^{(a-1)}$ for these a and $|\tau^{(a)}| = |\tau^{(a-1)}| + 1$ by Property 11. Otherwise, $|\tau^{(a)}| = |\tau^{(a-1)}|$ by definition of $\gamma^{(x-1)}/\rho^{(x)}$.

To prove Condition (asv3), note that all cells of $\gamma^{(x-1)}/\rho^{(x)}$ are removable corners of $\gamma^{(x-1)}$ since these cells contain both a letter larger and weakly smaller than $\Sigma^{x-1}\alpha$ and rows/columns are non-decreasing. We thus must show that if c and $\bar{c}$ are removable i-corners of $\gamma^{(x-1)}$ where $c \in \gamma^{(x-1)}/\rho^{(x)}$ and $\bar{c} \notin \gamma^{(x-1)}/\rho^{(x)}$, then $\bar{c}$ lies below a cell in $\gamma^{(x)}$.

Let $y = a + \Sigma^{x-1}\alpha$ denote the letter of $\mathcal{A}_{\alpha,x}$ in cell c of T. Then c is a removable i-corner of the core $\tau^{(a)}$ and Definition 7 implies there must be a y in all $\tau^{(a)}$-removable i-corners. Since $\bar{c}$ contains no letter of $\mathcal{A}_{\alpha,x}$, it is not $\tau^{(a)}$-removable. Therefore there is a letter $x \in \mathcal{A}_{\alpha,x}$ for some $x \leqslant y$ above or right-adj to $\bar{c}$. In fact, $x < y$ since y occupies only cells of residue i. Assuming the later case, $\bar{c}$ must lie weakly lower than c since Property 1 implies all i-extremals in $\tau^{(a)}$ that are higher than c must lie at the end of their row (and all removable corners of $\gamma^{(x-1)}$ are extremal in $\gamma^{(x)}$ by horizontality). If we choose c to be the cell containing the lowest y, there is a letter of $\mathcal{A}_{\alpha,x}$ that is smaller and weakly lower than this y contradicting that the lowest reading word of an affine s-v tableau is increasing. $\square$

5. Affine Grothendieck polynomials

Affine stable Grothendieck polynomials were introduced in [9] in terms of the nil Hecke algebra. Recall that the nil Hecke algebra K for the type-A affine Weyl group is generated over $\mathbb{Z}$ by $A_0, A_1, \ldots, A_k$ and relations

$$A_i^2 = -A_i \quad \text{for all } i, \qquad A_iA_j = A_jA_i \quad \text{if } |i-j| > 2, \qquad A_iA_{i+1}A_i = A_{i+1}A_iA_{i+1}$$

where the indices are taken modulo $k+1$ [7]. The algebra K is a free Z-module with basis $\{A_w\colon\ w \in \tilde{S}_{k+1}\}$ where $A_w = A_{i_1}\cdots A_{i_\ell}$ for any reduced word $i_1\cdots i_\ell$ of w. In this basis, the multiplication is given by

$$A_iA_u = \begin{cases} A_{s_iu} & \text{if } \ell(s_iu) > \ell(u), \\ -A_u & \text{if } \ell(s_iu) < \ell(u). \end{cases}$$

The definition of affine Grothendieck polynomials requires elements defined by cyclically decreasing permutations. To be precise, let $i_1\cdots i_\ell$ be a sequence of numbers where each $i_r \in [0,k]$.

$i_1 \cdots i_\ell$ is *cyclically decreasing* if no number is repeated and j precedes $j-1$ (taken modulo $k+1$) when both $j, j-1 \in \{i_1, \ldots, i_\ell\}$. If $i_1 \cdots i_\ell$ is cyclically decreasing then we say the permutation $w = s_{i_1} \cdots s_{i_\ell}$ is cyclically decreasing. Note that w is reduced and depends only on the set $\{i_1, \ldots, i_\ell\}$ of indices involved. This given, consider

$$h_i = \sum_{\substack{w \in \tilde{S}_{k+1}:\ \ell(w)=i \\ w \text{ cyclically decreasing}}} A_w.$$

Then, for any $w \in \tilde{S}_{k+1}$, the *affine stable Grothendieck polynomial* is defined by

$$G_w^{(k)}(x_1, x_2, \ldots) = \sum_\alpha \langle h_{\alpha_\ell} h_{\alpha_{\ell-1}} \cdots h_{\alpha_1}, A_w \rangle x^\alpha. \tag{32}$$

We can explicitly describe the coefficients in this expression using certain factorizations of permutations. Define an *α-factorization* of w to be a decomposition of the form $w = w^{\ell(\alpha)} \cdots w^1$ where w^i is a cyclically decreasing permutation of length α_i. From this viewpoint, the coefficient of A_w in

$$h_{\alpha_\ell} \cdots h_{\alpha_1} = \sum_{\substack{\ell(w^\ell)=\alpha_\ell \\ w^\ell \text{ cyclically dec}}} A_{w^\ell} \cdots \sum_{\substack{\ell(w^1)=\alpha_1 \\ w^1 \text{ cyclically dec}}} A_{w^1}$$

is the signed enumeration of α-factorizations of w. Therefore,

$$G_w^{(k)}(x_1, x_2, \ldots) = \sum_\alpha (-1)^{|\alpha|-\ell(w)} \sum_{\alpha\text{-factorizations of } w} x^\alpha. \tag{33}$$

We give a bijection between affine s-v tableaux of shape λ and weight α and *α-factorizations* of w_λ. From this, the stable affine Grothendieck polynomials indexed by Grassmannian permutations are none other than generating functions for affine s-v tableaux. One advantage of such an identification is that properties of the tableaux given in prior sections immediately reveal basic facts about affine Grothendieck polynomials.

Lemma 27. *Given a cyclically decreasing word $i_r \cdots i_1$, let $T = \mathfrak{s}_{i_r,1} \cdots \mathfrak{s}_{i_1,1}(\beta)$ for any core β. If the ones in T occupy r distinct residues, then $(\gamma/\beta, \rho)$ is an affine s-v strip for $\gamma = \mathrm{shape}(T)$ and ρ the shape obtained by deleting all ones from T.*

Proof. Since $\mathfrak{s}_{i_x-1,1}$ is never applied after $\mathfrak{s}_{i_x,1}$ by the definition of cyclically decreasing, γ/ρ is horizontal. Let $\beta^{(0)} = \beta$ and set $\beta^{(x)} = \mathfrak{s}_{i_x}(\beta^{(x-1)})$ for $x = 1, \ldots, r$. An element of β/ρ arises only when $\mathfrak{s}_{i_{x+1}}$ is applied to $\beta^{(x)}$ and there is $\beta^{(x)}$-removable i_{x+1}-corner. In this case, all i_{x+1}-corners of $\beta^{(x)}$ are β-removable since $\beta^{(x)}/\beta$ has no i_{x+1}-residue. Further, any cell c' right-adj to a β-removable i_{x+1}-corner c has residue $i_{x+1}+1$ and thus is not in $\beta^{(x)}$ by definition of cyclically decreasing. Therefore c is either $\beta^{(x)}$-removable (and in β/ρ) or it is γ-blocked.

It thus remains to prove that γ/β is an affine $r-m$ strip, where $m = |Res(\beta/\rho)|$. Since the ones in T occupy r distinct residues, $Res(\gamma/\rho) = \{i_1, \ldots, i_r\}$ and $\beta^{(x-1)}$ has a removable or an addable i_x-corner. Since $\{i_1, \ldots, i_r\}$ are distinct and a core never has both an addable and removable corner of the same residue, $Res(\beta^{(x)}/\beta) \cap Res(\beta/\rho) = \emptyset$. Therefore, by Remark 14(1), $|\mathfrak{p}(\gamma)| = |\mathfrak{p}(\beta)| + r - m$. $\square$

Theorem 28. *For $\lambda \in \mathcal{P}^k$ and k-bounded composition α, there is a bijection between $\mathcal{T}_\alpha^k(\lambda)$ and the set of α-factorizations for w_λ.*

Proof. ($\Leftarrow$): Let $\ell = \ell(\alpha)$ for a k-bounded composition α. Consider an α-factorization $w_\lambda = w^\ell w^{\ell-1} \cdots w^1$ and let $i_{\Sigma^x \alpha} \cdots i_{\Sigma^{x-1}\alpha+1}$ be a cyclically decreasing word for w^x, for each $x = 1, \ldots, \ell$. Starting from the empty tableau $U^{(0)}$, iteratively construct

$$U^{(x)} = \mathfrak{s}_{i_{\Sigma^x\alpha},x} \cdots \mathfrak{s}_{i_{\Sigma^{x-1}\alpha+1},x}(U^{(x-1)})$$

for $1 \leqslant x \leqslant \ell$. Let $\gamma^{(x)} = \text{shape}(U^{(x)})$ and let $\rho^{(x)}$ be the shape of $U^{(x)}$ minus its cells containing the letter x. By Theorem 26, it suffices to show that $(\gamma^{(x)}/\gamma^{(x-1)}, \rho^{(x)})$ is an affine s-v α_x-strip.

Since w_λ is Grassmannian and w^1 is cyclically decreasing, $w^1 = s_{\alpha_1-1} \cdots s_0$. Thus the result holds for $x = 1$ since $\gamma^{(1)} = \mathfrak{s}_{\alpha_1-1} \cdots \mathfrak{s}_0 \emptyset$ is horizontal and $\rho^{(1)} = \emptyset$. By induction, $\gamma^{(\ell-1)}$ is a core and by Remark 16, the letter ℓ in $U^{(\ell)}$ occupies α_ℓ distinct residues. The result then follows by applying Lemma 27 since $\gamma^{(\ell)} = \mathfrak{s}_{i_{\Sigma^\ell\alpha}} \cdots \mathfrak{s}_{i_{\Sigma^{\ell-1}\alpha+1}} \gamma^{(\ell-1)}$.

($\Rightarrow$): Given $T \in \mathcal{T}_\alpha^k(\lambda)$, let $w^x = s_{j_{\Sigma^x\alpha}} \cdots s_{j_{\Sigma^{x-1}\alpha+1}}$ where j_a denotes the residue of letter a in T, for $x = 1, \ldots, \ell(\alpha)$. Proposition 17 implies that $w_\lambda = w^\ell \cdots w^1$ and it remains to show that w^x is cyclically decreasing.

The letters of $\mathcal{A}_{\alpha,x}$ occupy residues $\mathcal{S} = \{j_{\Sigma^{x-1}\alpha+1}, \ldots, j_{\Sigma^x\alpha}\}$. The definition of affine s-v tableau implies residues in $\mathcal{S}$ are distinct, the lowest reading word of $\mathcal{A}_{\alpha,x}$ is increasing, and γ/ρ is horizontal, where $\gamma = \text{shape}(T_{\leqslant \Sigma^x\alpha})$ and ρ is the shape of $T_{\leqslant \Sigma^x\alpha}$ minus cells containing an element of $\mathcal{A}_{\alpha,x}$. Suppose $i, i-1 \in \mathcal{S}$ and $i-1$ precedes i in $j_{\Sigma^x\alpha} \cdots j_{\Sigma^{x-1}\alpha+1}$. Then there are letters t_1 and t_2 (of residues i and $i-1$, respectively) in $\mathcal{A}_{\alpha,x}$ where $t_1 < t_2$. Since the lowest reading word is increasing, the lowest t_2 occurs to the right of t_1 and they do not lie in the same row since $\alpha_x \leqslant k$. Further, there is no element of $\mathcal{A}_{\alpha,x}$ right-adj to t_2 since this could only be t_1. Therefore, all extremals of residue $i-1$ lie at the end of their row by Property 1. However, t_1 is right-adj to an extremal of residue $i-1$ by the horizontality of γ/ρ. $\square$

Corollary 29. *For any $\lambda \in \mathcal{P}^k$,*

$$G_\lambda^{(k)} = \sum_{T \in \mathcal{T}^k(\lambda)} (-1)^{|\lambda|+|\text{weight}(T)|} x^{\text{weight}(T)}, \tag{34}$$

where $G_\lambda^{(k)} = G_{w_\lambda}^{(k)}$.

This interpretation for the $G_\lambda^{(k)}$ allows us to establish a number of properties using our results on affine s-v tableaux. To start, Mark Shimozono conjectured that affine Grothendieck polynomials for the Grassmannian reduce to Grothendieck polynomials in limiting cases of k. In fact, we find precisely that:

Property 30. If $h(\lambda) \leqslant k$, then $G_\lambda^{(k)} = G_\lambda$.

Proof. Proposition 9 tells us that the elements of $\mathcal{T}^k(\lambda)$ are set-valued tableaux of shape λ when $h(\lambda) \leqslant k$. The result thus follows from Corollary 29 and the definition for Grothendieck polynomials. $\square$

It was shown in [9] that $G_w^{(k)}$ of (32) are symmetric functions. Thus, letting

$$\mathcal{K}_{\lambda\alpha}^{(k)} = \left|\mathcal{T}_\alpha^k(\lambda)\right|$$

enumerate the affine s-v tableaux, we deduce a symmetry of this affine K-theoretic refinement of the Kostka numbers from Corollary 29.

Corollary 31.[2] *Given any $\lambda \in \mathcal{P}^k$ and k-bounded composition α,*

$$\mathcal{K}_{\lambda\alpha}^{(k)} = \mathcal{K}_{\lambda\beta}^{(k)}$$

for any rearrangement β of α.

The affine Grothendieck polynomial can then be written as, for $\lambda \in \mathcal{P}^k$,

$$G_\lambda^{(k)} = \sum_{\mu\in\mathcal{P}^k} (-1)^{|\lambda|+|\mu|}\mathcal{K}_{\lambda\mu}^{(k)} m_\mu. \tag{35}$$

Our earlier result showing affine s-v tableaux are simply k-tableaux in certain cases also enables us to refine this expansion and connect affine Grothendieck polynomials to dual k-Schur functions.

Property 32. For any k-bounded partitions λ and μ,

$$\mathcal{K}_{\lambda\mu}^{(k)} = \begin{cases} 1 & \text{when } \mu = \lambda, \\ 0 & \text{when } |\mu| = |\lambda| \text{ and } \lambda \ntrianglerighteq \mu, \\ 0 & \text{when } |\mu| < |\lambda|. \end{cases} \tag{36}$$

Proof. Consider an affine s-v tableau T of shape $\mathfrak{c}(\lambda)$ and weight μ. We have that $|\lambda| \geqslant |\mu|$ by Corollary 12. Further, Proposition 13 implies that T is a k-tableau when $|\mu| = |\lambda|$, in which case the desired relation follows from (22). $\square$

We thus have the unitriangularity relation:

$$G_\lambda^{(k)} = m_\lambda + \sum_{\substack{\mu\in\mathcal{P}^k \\ \mu\lhd\lambda}} \mathcal{K}_{\lambda\mu}^{(k)} m_\mu + \sum_{\substack{\mu\in\mathcal{P}^k \\ |\mu|>|\lambda|}} (-1)^{|\lambda|+|\mu|}\mathcal{K}_{\lambda\mu}^{(k)} m_\mu, \tag{37}$$

and it follows immediately that these affine Grothendieck polynomials are a basis.

[2] A direct combinatorial proof of this symmetry will appear in [1] using an involution from the set of α-factorizations of w_λ to the set of $\hat{\alpha}$-factorizations, where $\hat{\alpha}$ is obtained by transposing two adjacent components of α. The involution generalizes the Lascoux–Schützenberger symmetric group action on words.

Property 33. $\{G_\lambda^{(k)}\}_{\lambda\in\mathcal{P}^k}$ is a Schauder basis for $\Lambda/\mathcal{I}^k$.

In analogy to (17), we also deduce that the dual k-Schur expansion of an affine Grothendieck polynomial has integer coefficients and the polynomial made up of the lowest homogeneous degree terms is precisely a dual k-Schur function.

Property 34. For any k-bounded partition λ,

$$G_\lambda^{(k)} = \mathfrak{S}_\lambda^{(k)} + \sum_{\substack{\mu\in\mathcal{P}^k\\ |\mu|>|\lambda|}} a_{\lambda\mu}^k \mathfrak{S}_\mu^{(k)} \quad \text{for } a_{\lambda\mu}^k \in \mathbb{Z}. \tag{38}$$

Proof. The bottom degree terms of expression (37) matches the monomial expansion (23) for the dual k-Schur function $\mathfrak{S}_\lambda^{(k)}$ since $\mathcal{K}_{\lambda\mu}^{(k)} = K_{\lambda\mu}^{(k)}$ when $|\mu| = |\lambda|$ by Proposition 13. The higher degree terms involve $m_\mu \in \Lambda/\mathcal{I}^k$ and can thus be expanded into the $\{\mathfrak{S}_\lambda^{(k)}\}$-basis. The coefficients remain integral by the unitriangularity of expansion (23). $\square$

On one hand, as their name suggests, the affine Grothendieck polynomials can be viewed as an affine analog of the Grothendieck polynomials. At a fundamental level, because the expansion coefficients in (17) are in fact positive (up to a degree-alternating sign), Thomas Lam conjectured the same about the coefficients $a_{\lambda\mu}^k$. In the same vein, it was proven in [24] that the coefficients in

$$s_\lambda = \sum_\mu f_{\lambda\mu} G_\mu \tag{39}$$

have a simple combinatorial interpretation as the number of certain restricted skew tableaux. Evidence suggests that the affine analog of this identity is also positive.

Conjecture 35. *For any k-bounded partition λ, the coefficients $f_{\lambda\mu}^k$ in*

$$\mathfrak{S}_\lambda^{(k)} = \sum_{\substack{\mu\in\mathcal{P}^k\\ |\mu|\geqslant|\lambda|}} f_{\lambda\mu}^k G_\mu^{(k)} \tag{40}$$

are non-negative integers.

On the other hand, viewing affine Grothendieck polynomials as the K-theoretic analog of dual k-Schur functions suggests that these polynomials satisfy even more refined combinatorial properties. For example, it is proven in the forthcoming paper [12] that the coefficients in

$$\mathfrak{S}_\lambda^{(k+1)} = \sum_{\mu\in\mathcal{P}^k} a_{\lambda,\mu}^{k+1,k} \mathfrak{S}_\mu^{(k)} \mod \mathcal{I}^k \tag{41}$$

are non-negative integers. Since a dual k-Schur function reduces to a Schur function for large k, this expression can be iterated to imply the positivity of coefficients in

$$s_\lambda = \sum_{\mu \in \mathcal{P}^k} a^k_{\lambda,\mu} \mathfrak{S}^{(k)}_\mu \mod \mathcal{I}^k, \tag{42}$$

for any $k > 0$. Naturally following suite in our setting leads to the K-theoretic version of these ideas.

Conjecture 36. *For any $k+1$-bounded partition λ, the coefficients $d^{k+1,k}_{\lambda\mu}$ in*

$$G^{(k+1)}_\lambda = \sum_{\substack{\mu \in \mathcal{P}^k \\ |\mu| \geqslant |\lambda|}} (-1)^{|\lambda|+|\mu|} d^{k+1,k}_{\lambda\mu} G^{(k)}_\mu \mod \mathcal{I}^k \tag{43}$$

are non-negative integers. By Property 30 this implies that the coefficients $d^k_{\lambda\mu}$ in

$$G_\lambda = \sum_{\substack{\mu \in \mathcal{P}^k \\ |\lambda| \leqslant |\mu|}} (-1)^{|\lambda|+|\mu|} d^k_{\lambda\mu} G^{(k)}_\mu \tag{44}$$

are non-negative integers.

Note that (44) is the analog of (42) where the Schur function s_λ is considered to be $\mathfrak{s}^{(\infty)}_\lambda$ and the dual k-Schur functions are then all replaced by their K-theoretic counterparts, the affine Grothendieck polynomials. Alternatively, if we do not interpret s_λ as a dual ∞-Schur auction, we can derive a conjecture about the affine Grothendieck expansion of a Schur function.

Conjecture 37. *For any k-bounded partition λ, the coefficients $b^k_{\lambda\mu}$ in*

$$s_\lambda = \sum_{\substack{\mu \in \mathcal{P}^k \\ |\mu| \geqslant |\lambda|}} b^k_{\lambda\mu} G^{(k)}_\mu \mod \mathcal{I}^k \tag{45}$$

are non-negative integers.

6. k-K-Schur functions

Thomas Lam conjectured in an FRG wiki post, and at the 2008 FRG problem solving session, that there is a basis $g^{(k)}_\lambda$ of $\Lambda^{(k)}$ such that:

(1) $\langle g^{(k)}_\lambda, G^{(k)}_\mu \rangle = \delta_{\lambda\mu}$;
(2) as $k \to \infty$, $g^{(k)}_\lambda$ reduces to the dual Grothendieck polynomial g_λ;
(3) the top homogeneous component of $g^{(k)}_\lambda$ is the k-Schur function $s^{(k)}_\lambda$;
(4) $g^{(k)}_\lambda$ can be expanded positively in terms of k-Schur functions.

It was with this in mind that we began the study of a second family of polynomials called *k-K-Schur functions*. Our point of departure is similar to what was done to define dual Grothendieck (18) and k-Schur functions (24), but now exploiting the invertibility of the matrix $\|\mathcal{K}^{(k)}\|_{\lambda,\mu \in \mathcal{P}^k}$ given by Property 32.

Definition 38. For every $\lambda \in \mathcal{P}^k$, the *k-K-Schur functions* $g_\lambda^{(k)}$ are defined by the system of equations,

$$h_\lambda = \sum_{\substack{\mu \in \mathcal{P}^k \\ |\mu| \leqslant |\lambda|}} (-1)^{|\lambda|+|\mu|} \mathcal{K}_{\mu\lambda}^{(k)} g_\mu^{(k)}. \tag{46}$$

In particular, we let $\|\bar{\mathcal{K}}_{\lambda\mu}^{(k)}\|_{\lambda,\mu\in\mathcal{P}^k}$ denote the inverse of $\|\mathcal{K}_{\lambda\mu}^{(k)}\|_{\lambda,\mu\in\mathcal{P}^k}$ and invert (46). The conditions on this matrix imposed by Property 32 imply

$$g_\lambda^{(k)} = h_\lambda + \sum_{\substack{\mu \in \mathcal{P}^k \\ \mu \rhd \lambda}} \bar{\mathcal{K}}_{\mu\lambda}^{(k)} h_\mu + \sum_{\substack{\mu \in \mathcal{P}^k \\ |\mu| < |\lambda|}} (-1)^{|\mu|+|\lambda|} \bar{\mathcal{K}}_{\mu\lambda}^{(k)} h_\mu. \tag{47}$$

From this, we extract a number of properties including proofs of conjectures (1)–(3).

Property 39. The set $\{g_\lambda^{(k)}\}_{\lambda_1 \leqslant k}$ forms a basis for $\Lambda^{(k)}$.

Property 40. For all $\lambda, \mu \in \mathcal{P}^k$,

$$\langle g_\lambda^{(k)}, G_\mu^{(k)} \rangle = \delta_{\lambda\mu}.$$

Proof. From Eq. (47) for k-K-Schur functions, we have that

$$\langle G_\mu^{(k)}, g_\lambda^{(k)} \rangle = \left\langle \sum_{|\mu| \leqslant |\beta|} (-1)^{|\mu|+|\beta|} \mathcal{K}_{\mu\beta}^{(k)} m_\beta, \sum_{|\alpha| \leqslant |\lambda|} (-1)^{|\lambda|+|\alpha|} \bar{\mathcal{K}}_{\alpha\lambda}^{(k)} h_\alpha \right\rangle,$$

implying

$$\langle G_\mu^{(k)}, g_\lambda^{(k)} \rangle = (-1)^{|\lambda|+|\mu|} \sum_\beta \mathcal{K}_{\mu\beta}^{(k)} \bar{\mathcal{K}}_{\beta\lambda}^{(k)} = \delta_{\lambda\mu}$$

by the duality of $\{m_\beta\}$ and $\{h_\alpha\}$. □

We have seen that the term of lowest degree in the affine Grothendieck polynomial is a dual k-Schur function. The affine analog of (19) is that the highest degree term of a k-K-Schur function is a k-Schur function. We also find that their expansion coefficients in terms of k-Schur functions and the dual Grothendieck polynomials are integers.

Property 41. For any k-bounded partition λ,

$$g_\lambda^{(k)} = s_\lambda^{(k)} + \sum_{\substack{\mu \in \mathcal{P}^k \\ |\mu| < |\lambda|}} f_{\lambda\mu}^k s_\mu^{(k)} \quad \text{for } f_{\lambda\mu}^k \in \mathbb{Z}. \tag{48}$$

Proof. Proposition 13 implies that $\mathcal{K}^{(k)}_{\mu\lambda} = K^{(k)}_{\mu\lambda}$ for all $|\lambda| = |\mu|$. Making this replacement in Eq. (47) gives that

$$g^{(k)}_\lambda = s^{(k)}_\lambda + \sum_{\substack{\mu\in\mathcal{P}^k\\ |\mu|<|\lambda|}} (-1)^{|\mu|+|\lambda|}\bar{\mathcal{K}}^{(k)}_{\mu\lambda} h_\mu .$$

We can then expand h_μ in terms of k-Schur functions using (24) and since $\bar{\mathcal{K}}^{(k)}_{\mu\lambda} \in \mathbb{Z}$ by unitriangularity, we find that the expansion coefficients of (48) are integers. □

The long-standing conjecture that k-Schur functions are Schur positive,

$$s^{(k)}_\lambda = \sum_\mu b^k_{\lambda\mu} s_\mu \quad \text{where } b^k_{\lambda\mu} \in \mathbb{N}, \tag{49}$$

was recently proven in [12] by establishing the more refined property that

$$s^{(k)}_\lambda = \sum_{\mu\in\mathcal{P}^{k+1}} b^{k,k+1}_{\lambda\mu} s^{(k+1)}_\mu \quad \text{where } b^{k,k+1}_{\lambda\mu} \in \mathbb{N}. \tag{50}$$

The Schur positivity follows from this because a k-Schur function reduces to a Schur function for large k.

We conjecture that the theory of k-K-Schur functions follows a similar path. To be precise, the homogeneous symmetric functions that arise in (47) can be integrally expanded in terms of the dual Grothendieck polynomials by (18).

Property 42. For any k-bounded partition λ,

$$g^{(k)}_\lambda = g_\lambda + \sum_{\mu\rhd\lambda} b_{\lambda\mu} g_\mu + \sum_{|\mu|<|\lambda|} b_{\lambda\mu} g_\mu \quad \text{for } b_{\lambda\mu} \in \mathbb{Z}. \tag{51}$$

Similarly, the homogeneous symmetric functions that arise in (47) can be instead be expanded integrally in terms of the $k+1$-K-Schur functions using Definition 38.

Property 43. For any $\lambda \in \mathcal{P}^k$,

$$g^{(k)}_\lambda = g^{(k+1)}_\lambda + \sum_{\mu\rhd\lambda} b^k_{\lambda\mu} g^{(k+1)}_\mu + \sum_{|\mu|<|\lambda|} b^k_{\lambda\mu} g^{(k+1)}_\mu \quad \text{for } b^k_{\lambda\mu} \in \mathbb{Z}. \tag{52}$$

Conjecture 44. *For all $\lambda, \mu \in \mathcal{P}^k$, the integer coefficient $(-1)^{|\lambda|+|\mu|} b^k_{\lambda\mu}$ is non-negative.*

Then, when k is large, a k-K-Schur function reduces simply to a dual Grothendieck polynomial.

Property 45. If $|\lambda| \leqslant k$, then $g^{(k)}_\lambda = g_\lambda$.

Proof. Proposition 9 implies that $\mathcal{K}^{(k)}_{\lambda\mu} = \mathcal{K}_{\lambda\mu}$ when $h(\lambda) \leqslant k$. In particular, if $|\lambda| \leqslant k$ then all partitions μ where $|\mu| \leqslant |\lambda|$ have k-bounded hook-length. □

From this, iterating Property 43 will eventually lead to the positive (up to alternating sign) expansion coefficients in (51) of a k-K-Schur function in terms of dual Grothendieck polynomials.

Conjecture 46. *For all $\lambda, \mu \in \mathcal{P}^k$, the integer coefficient $(-1)^{|\lambda|+|\mu|} b_{\lambda\mu}$ is non-negative.*

Property 47. For any partition λ with $h(\lambda) \leqslant k$, we have that

$$g^{(k)}_\lambda = g_\lambda + \text{lower degree terms}. \tag{53}$$

Note: we conjecture that all the lower degree terms cancel.

Proof. When $k \geqslant h(\lambda)$, it was shown in [17] that $s^{(k)}_\lambda = s_\lambda$. Thus, the k-Schur expansion of Property 41 reduces in this case to

$$g^{(k)}_\lambda = s_\lambda + \text{lower degree terms}. \tag{54}$$

The result then follows from (19) expressing g_λ as s_λ plus lower degree terms. □

7. Pieri rules

In addition to the basic properties of k-K-Schur functions extracted from the definition, we have also determined explicit Pieri rules for these polynomials.

7.1. Row Pieri rule

Theorem 48. *For any k-bounded partition λ and $r \leqslant k$,*

$$g^{(k)}_r g^{(k)}_\lambda = \sum_{(\mu,\rho)\in\mathcal{H}^k_{\lambda,r}} (-1)^{|\lambda|+r-|\mu|} g^{(k)}_\mu, \tag{55}$$

where $\mathcal{H}^k_{\lambda,r} = \{(\mu,\rho)\colon\ (\mathfrak{c}(\mu)/\mathfrak{c}(\lambda), \rho) = \text{affine set-valued } r\text{-strip}\}$.

Example 49.

$$g^{(3)}_2 g^{(3)}_{3,2,1} = g^{(3)}_{3,2,2,1} + g^{(3)}_{3,3,1,1} - g^{(3)}_{3,2,1,1} - 2g^{(3)}_{3,2,2} + g^{(3)}_{3,2,1}.$$

Proof. Note that $g^{(k)}_\ell = h_\ell$. Since the k-K-Schur functions form a basis of $\Lambda^{(k)}$, there is an expansion

$$h_\ell g^{(k)}_\nu = \sum_\mu c_{\mu\nu} g^{(k)}_\mu, \tag{56}$$

for some coefficients $c_{\mu\nu}$. To determine the $c_{\mu\nu}$, we examine $h_\ell h_\lambda$. Using the k-K-Schur expansion (46) for h_λ, we find that

$$h_\ell h_\lambda = \sum_\nu \mathcal{K}^{(k)}_{\nu\lambda} h_\ell g^{(k)}_\nu = \sum_\nu \mathcal{K}^{(k)}_{\nu\lambda} \sum_\mu c_{\mu\nu} g^{(k)}_\mu. \tag{57}$$

On the other hand, we can use (46) to expand $h_\ell h_\lambda = h_\tau$, where τ is the partition rearrangement of (ℓ, λ):

$$h_\ell h_\lambda = h_\tau = \sum_\mu \mathcal{K}^{(k)}_{\mu\tau} g^{(k)}_\mu = \sum_\mu \mathcal{K}^{(k)}_{\mu(\lambda,\ell)} g^{(k)}_\mu, \tag{58}$$

where the last equality holds by Corollary 31. Then, since Theorem 26 implies

$$\mathcal{K}^{(k)}_{\mu(\lambda,\ell)} = \sum_{\nu:\ \mu\in\mathcal{H}^k_{\nu,\ell}} \mathcal{K}^{(k)}_{\nu\lambda}, \tag{59}$$

we have

$$h_\ell h_\lambda = \sum_\mu \sum_{\nu:\ \mu\in\mathcal{H}^{(k)}_{\nu,\ell}} \mathcal{K}^{(k)}_{\nu\lambda} g^{(k)}_\mu. \tag{60}$$

Equating the coefficient of $g^{(k)}_\mu$ in the right side of this expression to that of (57) to get the system:

$$\sum_{\nu:\ \mu\in\mathcal{H}^{(k)}_{\nu,\ell}} \mathcal{K}^{(k)}_{\nu\lambda} = \sum_\nu \mathcal{K}^{(k)}_{\nu\lambda} c_{\mu\nu}. \tag{61}$$

We thus find our desired solution

$$c_{\mu\nu} = \left\{\begin{matrix} 1 & \text{if } \mu\in\mathcal{H}^{(k)}_{\nu,\ell} \\ 0 & \text{otherwise} \end{matrix}\right\}. \tag{62}$$

It is unique since any other solution satisfies

$$\sum_\nu \mathcal{K}^{(k)}_{\nu\lambda} (\tilde{c}_{\mu\nu} - c_{\mu\nu}) = 0, \tag{63}$$

and the invertibility of $\mathcal{K}^{(k)}_{\nu\lambda}$ implies $\tilde{c}_{\mu\nu} = c_{\mu\nu}$. $\square$

The Pieri rule can equivalently be phrased in the notation of affine permutations. In particular, Theorem 28 identifies affine s-v strips with cyclically decreasing words and we know (e.g. Remark 15) that $w_{(r)} = s_{r-1}\cdots s_0$.

Corollary 50. *For any $w \in \tilde{S}^0_{k+1}$ and $r \leqslant k$,*

$$g^{(k)}_{s_{r-1}\cdots s_0} g^{(k)}_w = \sum_{\substack{v=uw:\ \ell(u)=r \\ u \text{ cyclically decreasing}}} (-1)^{\ell(w)+r-\ell(v)} g^{(k)}_v. \tag{64}$$

Note that the highest degree terms in the r.h.s. of (55) are simply the terms given by the Pieri rule [17] for k-Schur functions:

$$h_r g^{(k)}_\nu = h_r s^{(k)}_\nu + \text{lower terms},$$

obtained by adding affine r-strips to $\mathfrak{c}(\nu)$.

7.2. Column Pieri rule

There is also a combinatorial rule to compute the k-K-Schur function expansion of $g^{(k)}_{1^\ell} g^{(k)}_\nu$ in terms of vertical strips rather than horizontal. The dual Grothendieck polynomial indexed by a column is

$$g_{1^\ell} = \sum_{j=1}^{\ell} \binom{\ell-1}{j-1} e_j, \tag{65}$$

and $g^{(k)}_{1^\ell} = g_{1^\ell}$ when $\ell \leqslant k$ by Property 45. To determine the associated Pieri rule, we start with a "K-theoretic" version of Newton's formula (e.g. [25]):

$$\sum_{r=0}^{\ell} (-1)^r h_{\ell-r} e_r = 0. \tag{66}$$

Proposition 51. *For any integer $\ell \geqslant 0$,*

$$\sum_{r=0}^{\ell} \sum_{j=0}^{r} (-1)^{j+r} \binom{r-2}{j} g_{\ell-r} g_{1^{r-j}} = 0. \tag{67}$$

Proof. By expression (65) for g_{1^ℓ}, this follows from the identity

$$\sum_{r=0}^{\ell} \sum_{j=0}^{r} \sum_{t=1}^{r-j} (-1)^{j+r} \binom{r-2}{j} \binom{r-j-1}{t-1} h_{\ell-r} e_t = 0. \tag{68}$$

Equivalently,

$$\sum_{r=0}^{\ell} \sum_{t=1}^{r} (-1)^r \left(\sum_{j=0}^{r-t} (-1)^j \binom{r-2}{j} \binom{r-j-1}{t-1} \right) h_{\ell-r} e_t = 0. \tag{69}$$

In fact, the orthogonality identity implies that

$$\sum_{j=0}^{r-t}(-1)^j\binom{r-2}{r-2-j}\left(\binom{r-j-2}{t-1}+\binom{r-j-2}{t-2}\right) \tag{70}$$

$$=(-1)^{r+t-1}\delta_{r-2,t-1}+(-1)^{r-t}\delta_{r-2,t-2} \tag{71}$$

and thus the l.h.s. of (69) reduces to

$$\sum_{r=0}^{\ell}\sum_{t=1}^{r}(-1)^r\big((-1)^{r+t-1}\delta_{r-2,t-1}+(-1)^{r-t}\delta_{r-2,t-2}\big)h_{\ell-r}e_t \tag{72}$$

$$=\sum_{r=0}^{\ell}(-1)^r h_{\ell-r}e_{r-1}+\sum_{r=0}^{\ell}(-1)^r h_{\ell-r}e_r \tag{73}$$

which vanishes by Newton's identity. □

Theorem 52. *For any k-bounded partition λ and integer $r \leqslant k$,*

$$g_{1^r}^{(k)}g_\lambda^{(k)}=\sum_{(\mu,\rho)\in\mathcal{E}_{\lambda,r}^k}(-1)^{|\lambda|+r-|\mu|}g_\mu^{(k)}, \tag{74}$$

where $(\mu,\rho)\in\mathcal{E}_{\lambda,r}^k$ iff $(\mu^{\omega_k},\rho')\in\mathcal{H}_{\lambda^{\omega_k},r}^k$.

Example 53.

$$g_{1,1}^{(3)}g_{3,2,1}^{(3)}=g_{3,2,1,1,1}^{(3)}+g_{3,2,2,1}^{(3)}-g_{3,2,1,1}^{(3)}-g_{3,2,2}^{(3)}+g_{3,2,1}^{(3)}.$$

Proof. Since $g_1=h_1$, Theorem 48 implies the case when $r=1$ and we assume by induction that the action of g_{1^s} for all $s<r$ is given by (74). To prove our assertion for multiplication by g_{1^r}, since Proposition 51 can be rewritten as

$$\sum_{s=0}^{r-1}\sum_{j=0}^{r-s}(-1)^s\binom{s+j-2}{j}g_{r-s-j}g_{1^s}+(-1)^r g_{1^r}=0, \tag{75}$$

it suffices to show

$$\sum_{s=0}^{r-1}\sum_{j=0}^{r-s}(-1)^s\binom{s+j-2}{j}g_{r-s-j}g_{1^s}g_\lambda^{(k)}+\sum_{(\mu^{\omega_k},\rho)\in\mathcal{H}_{\lambda^{\omega_k},r}^{(k)}}(-1)^{|\mu|-|\lambda|}g_\mu^{(k)}=0. \tag{76}$$

We claim that the coefficient of $g_\nu^{(k)}$ in the left side of this expression is zero for any $\nu\in\mathcal{P}^k$.

By induction, for $s<r$, the coefficient of $g_\nu^{(k)}$ in $g_{r-s-j}g_{1^s}g_\lambda^{(k)}$ is $(-1)^{|\nu|-|\lambda|-r+j}$ times the number of *vh-fillings* with weight $(s,r-s-j)$ defined by:

(i) letter x lies in cells of $\mathfrak{c}(\mu)/\rho$ where $(\mu^{\omega_k},\rho')\in\mathcal{H}_{\lambda^{\omega_k},s}^k$,
(ii) letter y lies in cells of $\mathfrak{c}(\nu)/\tau$ where $(\nu,\tau)\in\mathcal{H}_{\mu,r-s-j}^k$.

Denote the set of such fillings by $\mathcal{VH}^k_{s,r-s-j}(\nu,\lambda)$. This given, for fixed $\nu,\lambda\in\mathcal{P}^k$, (76) follows by proving the identity

$$\sum_{s=0}^{r}\sum_{j=0}^{r-s}(-1)^{r-s-j}\binom{s+j-2}{j}\left|\mathcal{VH}^k_{s,r-s-j}(\nu,\lambda)\right|=0. \tag{77}$$

We take a combinatorial approach. First we rewrite this expression as a single sum by introducing ordered multisets of signed vh-fillings to account for the binomial numbers and then describe a sign-reversing involution to achieve cancellation. The desired involution will act by permuting certain "*free*" entries of $T\in\mathcal{VH}^k_{(a,b)}(\nu,\lambda)$, defined:

- $\{x,y\}_i$ is free if every x and y in $T_{\downarrow i}$ share a cell,
- $\{x\}_i$ is free if every $x\in T_{\downarrow i}$ occurs alone and at the top of its column,
- $\{y\}_i$ is free if every $y\in T_{\downarrow i}$ occurs alone and is not right-adj to an x or y.

Recall that the subscript on entry X in T is the residue of the cell containing X.

Let $a=\binom{s+j-2}{j}$, $b=\binom{s+j-3}{j}$, and $c=\binom{s+j-3}{j-1}$. For $T\in\mathcal{VH}^k_{s,r-s-j}(\nu,\lambda)$, let $\mathcal{S}_T$ be the multiset containing $|a|$ copies of $(\mathrm{sign}(a),T)$ if the lowest free entry is not $\{x\}$. Otherwise, let $\mathcal{S}_T$ be the ordered multiset with $|b|$ copies of $(\mathrm{sign}(b),T)$ followed by $|c|$ copies of $(\mathrm{sign}(c),T)$. Eq. (77) is then

$$\sum_{s=0}^{r}\sum_{j=0}^{r-s}(-1)^{r-s-j}\sum_{T\in\mathcal{VH}^k_{s,r-s-j}(\nu,\lambda)}\ \sum_{(\sigma,T)\in\mathcal{S}_T}\sigma=0. \tag{78}$$

If $\mathcal{T}^k_r(\nu,\lambda)$ is the union of multisets $\mathcal{S}_T$, for all $T\in\mathcal{VH}^k_{s,r-j-s}(\nu,\lambda)$ where $j,s\geqslant 0$ and $0\leqslant j+s\leqslant r$, then the above expression reduces to

$$\sum_{(\sigma,T)\in\mathcal{T}^k_r(\nu,\lambda)}(-1)^{\mathrm{weight}_y(T)}\times\sigma=0, \tag{79}$$

where $\mathrm{weight}_y(T)$ is the number of residues occupied by y's. Our result will follow from Property 58 which gives an involution $\mathfrak{m}$ on $\mathcal{T}^k_r(\nu,\lambda)$ where $\mathfrak{m}(\sigma,T)=(\sigma,\hat{T})$ with the property that $\mathrm{weight}_y(T)=\mathrm{weight}_y(\hat{T})\pm 1$. $\square$

Definition 54. Define the map

$$\mathfrak{m}:\mathcal{T}^k_r(\nu,\lambda)\to\mathcal{T}^k_r(\nu,\lambda)$$

on (σ,T) in position p of $\mathcal{S}_T$ as follows: let s,j,m,i be integers where $\mathrm{weight}(T)=(s,r-s-j)$ and the lowest free entry of T has residue i and lies in row m. Then $\mathfrak{m}(\sigma,T)=(\sigma,\hat{T})$ is in position $\hat{p}$ where

(1) if row m contains a free $\{y\}_i$, then $\hat{p}=p$ and $\hat{T}$ is obtained by replacing all $\{y\}_i$ in T by $\{x\}$,
(2) if row m contains a free $\{x,y\}_i$, then $\hat{p}=|\binom{s+j-2}{j+1}|+p$ and $\hat{T}$ is obtained by replacing all $\{x,y\}_i$ in T by $\{x\}$,

(3) if row m contains a free $\{x\}_i$, then
   (a) if $p \leqslant |\binom{s+j-3}{j}|$ then $\hat{p} = p$ and $\hat{T}$ is obtained by replacing all $\{x\}_i$ in T by $\{y\}$,
   (b) otherwise $\hat{p} = p - |\binom{s+j-3}{j}|$ and $\hat{T}$ is obtained by replacing all $\{x\}_i$ in T by $\{x, y\}$.

Let us emphasize that a vh-filling $T \in \mathcal{VH}^k_{(a,b)}(\nu, \lambda)$ is constructed as follows: take the transpose of the tableau obtained by putting x's in an affine s-v a-strip added to $\mathfrak{c}(\lambda)'$. To the resulting tableau, add an affine s-v b-strip filled with y's.

Lemma 55. *Given a vh-filling T, if there is an $\{x, y\}_i \in T$ that is not free then there must be an x or a y in $T_{\downarrow i-1}$.*

Proof. Consider $T \in \mathcal{VH}^k_{(a,b)}(\nu, \lambda)$ with a cell c containing a non-free $\{x, y\}_i$. Given x lies in cells of $\mathfrak{c}(\mu)/\rho$ where $(\mu^{\omega_k}, \rho') \in \mathcal{H}^k_{\lambda^{\omega_k},a}$, we have that $\beta = \mathfrak{c}(\mu)$ is the shape obtained by deleting all lonely y's from T. Thus c is β-removable since no x lies above a y and the x's form a vertical strip.

If c' contains a $\{y\}_i$ then $c' \notin \beta$ is above a cell in β since the y's form a horizontal strip. Therefore the cell left-adj to c' (of residue $i - 1$) contains a y since β cannot have an addable and removable i-corner. On the other hand, if c' contains an $\{x\}_i$, assume $x \notin T_{\downarrow i-1}$. Then c' is at the top of its column in β and is thus β-removable. Further, $c \in \beta/\rho$ where ρ is the shape obtained by deleting from T any cell containing a y. Thus, all non-blocked β-removable i-corners are in β/ρ implying that c' is blocked (by a y of residue $i - 1$). $\square$

Property 56. $\mathfrak{m}$ is well-defined.

Proof. Consider $(\sigma, T) \in \mathcal{T}^k_r(\nu, \lambda)$. By definition of free, no row of T contains more than one free entry since x's form a vertical strip in vh-fillings. It thus suffices to show that T contains a free $\{x\}$, a free $\{y\}$, or a free $\{x, y\}$. Suppose no entries are free. An arbitrary letter x or y lies in $T_{\downarrow i}$, for some i, and thus $T_{\downarrow i}$ has an entry $\{x\}$, $\{y\}$ or $\{x, y\}$. If $\{x\}_i$ or $\{y\}_i$ is in T, then it is not free implies there is an $\{x, y\}_i$, or some letter x or y of residue $i - 1$ in T. On the other hand, if there is an $\{x, y\}_i \in T$, then Lemma 55 implies there is a letter x or a y in $T_{\downarrow i-1}$. Therefore, there must be an x or a y in $T_{\downarrow i-1}$. From this, the same argument implies there must be an x or a y in $T_{\downarrow i-2}$. By iteration, T contains the letters $z_{\downarrow i}, z_{\downarrow i-1}, z_{\downarrow i-2}, \ldots, z_{\downarrow i+2}, z_{\downarrow i+1}$, where each $z_{\downarrow t}$ is x or y of residue t. This contradicts that T has weight $(s, r - j - s)$ for $r - j \leqslant k$. $\square$

Lemma 57. *Given $T \in \mathcal{T}^k_r(\nu, \lambda)$*

(1) *if $\{x\}_i$ is the lowest free entry in T then $y \notin T_{\downarrow i}$,*
(2) *if $\{y\}_i$ is the lowest free entry in T then $x \notin T_{\downarrow i}$.*

Proof. Given $T \in \mathcal{VH}^k_{(a,b)}(\nu, \lambda)$, x lies in cells of $\mathfrak{c}(\mu)/\rho$ where $(\mu^{\omega_k}, \rho') \in \mathcal{H}^k_{\lambda^{\omega_k},a}$ and letter y lies in cells of $\mathfrak{c}(\nu)/\tau$ where $(\nu, \tau) \in \mathcal{H}^k_{\mu,b}$.

(1): Any free $\{x\}_i$ is a removable corner of $\mathfrak{c}(\mu)$ since it lies at the top of its column and the x's form a vertical strip. Therefore, by Property 21, there can be no y of residue i in the affine strip $\mathfrak{c}(\nu)/\mathfrak{c}(\mu)$. Suppose $y \in \mathfrak{c}(\mu)_{\downarrow i}$. Then there is a y in all removable i-corners of $\mathfrak{c}(\mu)$ that are not

$\mathfrak{c}(\nu)$-blocked. Since the free $\{x\}_i$ is not blocked, it shares a cell with y, violating the definition of free.

(2): Given the lowest free entry is a $\{y\}_i$ in cell c_y of row m, suppose there is an $x \in T_{\downarrow i}$ in cell c_x of row m_x. Since $\{y\}_i$ is free, c_x contains a *lonely* $\{x\}_i$. In $\mathfrak{c}(\mu)$, cell c_x lies above an $i+1$-extremal by verticality of x's and the cell beneath c_y is at the top of its column by horizontality of y's. Therefore, by Property 1, $m_x > m$. Let β be the shape obtained by deleting from T all lonely y's *and* all lonely x's that have a residue $j \in B$, where B is the set of residues labeling any x that lies above the highest x of residue i. Proposition 25 implies that β is a core and $c_x(i)$ lies at the top of its column. However, the $\{y\}$ in cell c_y is free and thus is right-adj to a cell in β that lies above an i-extremal. We reach a contradiction by Property 1. □

Property 58. The map $\mathfrak{m}$ is an involution on $\mathcal{T}_r^k(\nu,\lambda)$ and for $\mathfrak{m}(\sigma,T) = (\sigma,\hat{T})$, $\text{weight}_y(T) = \text{weight}_y(\hat{T}) \pm 1$.

Proof. Given $(\sigma,T) \in \mathcal{T}_r^k(\nu,\lambda)$, define p, r, s so that (σ,T) is in position p of $\mathcal{S}_T$ and $\text{weight}(T) = (s, r-j-s)$. We will show that $\mathfrak{m}(\sigma,T) = (\sigma,T_1) \in \mathcal{T}_r^k(\nu,\lambda)$, $\text{weight}_y(T_1) = r-j-s\pm 1$, and $\mathfrak{m}^2 = 1$. Let m denote the lowest row with a free entry in T and set $a = \binom{s+j-2}{j}$, $b = \binom{s+j-3}{j}$, and $c = \binom{s+j-3}{j-1}$.

Consider the case when row m of T has a free $\{y\}_i$. Then $p \leqslant |\mathcal{S}_T| = |a|$ and $\sigma = \text{sign}(a)$. In this case, T_1 is obtained by replacing each $\{y\}_i$ in T by $\{x\}$. Any $y \in T_{\downarrow i}$ is lonely by definition of free and $x \notin T_{\downarrow i}$ by Lemma 57. Therefore, any x in $T_{1\downarrow i}$ is lonely and lies at the top of its column by horizontality of y's. From this, the lowest free entry in T_1 is the $\{x\}_i$ in row m and the weight of T_1 is $(s+1, r-j-s-1)$. Thus $\mathcal{S}_{T_1}$ has $(\text{sign}(a), T_1)$ in its first $|a|$ positions by definition of $\mathcal{S}_{T_1}$. In particular, there is a (σ,T_1) in position $\hat{p} = p \leqslant |a|$ of $\mathcal{S}_{T_1}$. Moreover, since the lowest free in T_1 is an $\{x\}_i$ and $y \notin T_{1\downarrow i}$, $\mathfrak{m}$ acts on (σ,T_1) by replacing each $\{x\}_i$ by $\{y\}$ and the (σ,T) in position p is recovered.

In the case that row m of T contains a free $\{x,y\}_i$, we again have $\sigma = \text{sign}(a)$ and $p \leqslant |a|$. T_1 is obtained by replacing each $\{x,y\}_i$ in T by $\{x\}$. The definition of free implies there are no lonely x or y in $T_{\downarrow i}$ and therefore T_1 has weight $(s, r-j-s-1)$. Further, the lowest free entry in T_1 is an $\{x\}_i$ in row m since each $\{x,y\}_i$ in T lies at the top of its column by horizontality of y's and is sent to a lonely x in $T_{1\downarrow i}$. This given, there are $|\binom{s+j-2}{j+1}| + |a|$ elements of $\mathcal{S}_{T_1}$ of which the last $|a|$ entries are $(\text{sign}(a), T_1)$. Thus, $p \leqslant |a|$ implies $(\text{sign}(a), T_1)$ is in position $\hat{p} = p + |\binom{s+j-2}{j+1}|$. Further, $\mathfrak{m}$ acts by replacing each $\{x\}_i$ in T_1 by $\{x,y\}$ and we have $\mathfrak{m}^2 = id$.

The last case is when there is a free $\{x\}_i$ in row m. There are two scenarios depending on p. When $p \leqslant |b|$, $\sigma = \text{sign}(b)$ and T_1 is obtained by replacing $\{x\}_i$ with $\{y\}$ in T. Since $y \notin T_{\downarrow i}$ by Lemma 57, the weight of T_1 is $(s-1, r-j-s+1)$. Further, the lowest free in T_1 is $\{y\}_i$ in row m since there is at most one x in each row of T implies that no x or y is left-adj to $\{y\}_i$ in T_1. Thus, the $|b|$ entries of $\mathcal{S}_{T_1}$ are $(\text{sign}(b), T_1)$. Therefore, there is a (σ,T_1) in position $\hat{p} = p$ of $\mathcal{S}_{T_1}$. When $\mathfrak{m}$ acts on (σ,T_1), each $\{y\}_i$ in T_1 is replaced by $\{x\}$ and we recover (σ,T) in position p.

Otherwise, $|b|+1 \leqslant p \leqslant |b|+|c|$ and $\sigma = \text{sign}(c)$. T_1 is obtained by replacing $\{x\}_i$ by $\{x,y\}$. Since $y \notin T_{\downarrow i}$ by Lemma 57, the weight of T_1 is $(s, r-j-s+1)$ and every x of residue i lies with y and vice versa. Thus the lowest free entry is an $\{x,y\}_i$ in row m implying that $\mathcal{S}_{T_1}$ is $|c|$ copies of $(\text{sign}(c), T_1)$. Therefore, there is a (σ,T_1) in position $p-|b| \leqslant |c|$ of $\mathcal{S}_{T_1}$. When $\mathfrak{m}$ acts on (σ,T_1), each $\{x,y\}_i$ in T_1 is replaced by $\{x\}$ and we recover (σ,T) in position p. □

8. Conjugating affine Grothendieck polynomials

An important property in the theory of Schur functions and k-Schur functions involves the algebra automorphism defined on Λ by $\omega : h_\ell \to e_\ell$. Not only does ω send s_λ to the single Schur function $s_{\lambda'}$, but it was proven in [17] that

$$\omega\big(s_\lambda^{(k)}\big) = s_{\lambda^{\omega_k}}^{(k)}. \tag{80}$$

In our study, we consider the algebra endomorphism defined on Λ by

$$\Omega h_\ell = \sum_{j=1}^{\ell} \binom{\ell-1}{j-1} e_j \tag{81}$$

to be an inhomogeneous analog of ω. Note: the transformation $e_\ell \to \sum_{j=1}^{\ell} \binom{\ell-1}{j-1} e_j$ has been studied [2] and is needed to relate the cohomology ring to the Grothendieck ring. In fact, the polynomials $\sum_{j=1}^{\ell} \binom{\ell-1}{j-1} e_j$ are connected to the study of classes of a Schubert subvariety of the Grassmannian in these rings [19].

Remark 59. A manipulatorial proof that Ω is an involution on $\Lambda^{(k)}$, supplied by Adriano Garsia, shows that

$$\begin{aligned}
\sum_{\ell\geqslant 1}\left(\frac{u}{u-1}\right)^{\ell}\Omega h_\ell &= \sum_{\ell\geqslant 1}\left(\frac{u}{u-1}\right)^{\ell}\sum_{j=1}^{\ell}\binom{\ell-1}{j-1}e_j\\
&= \sum_{j\geqslant 1} e_j \sum_{\ell\geqslant j}\binom{\ell-1}{j-1}\left(\frac{u}{u-1}\right)^{\ell-j}\left(\frac{u}{u-1}\right)^{j}\\
&= \sum_{j\geqslant 1} e_j \frac{(\frac{u}{u-1})^j}{1-(\frac{u}{u-1})^j} = \sum_{j\geqslant 1}(-1)^j e_j u^j
\end{aligned}$$

implies by Newton's formula (66) that

$$\left(\sum_{\ell\geqslant 0} u^\ell h_\ell\right)\left(\sum_{\ell\geqslant 0}\left(\frac{u}{u-1}\right)^{\ell}\Omega h_\ell\right) = 1.$$

The result follows by substituting $u = \frac{u}{u-1}$ into this expression and applying Ω:

$$\left(\sum_{\ell\geqslant 0}\left(\frac{u}{u-1}\right)^{\ell}\Omega h_\ell\right)\left(\sum_{\ell\geqslant 0} u^\ell \Omega^2 h_\ell\right) = 1.$$

Remark 60. By Jacobi–Trudi we have

$$\Omega(s_\lambda) = s_{\lambda'} + \text{lower degree terms}.$$

This involution acts beautifully on the k-K-Schur functions just as ω acts on a Schur function (or more generally, a k-Schur function).

Theorem 61. *For any k-bounded partition λ,*

$$\Omega g_\lambda^{(k)} = g_{\lambda^{\omega_k}}^{(k)}. \tag{82}$$

Proof. Let $F_\lambda = \Omega(g_{\lambda^{\omega_k}}^{(k)})$. Since $h_\ell \Omega(g_\lambda^{(k)}) = \Omega(\sum_{j=1}^{\ell} \binom{\ell-1}{j-1} e_j g_\lambda^{(k)})$, we can apply the column Pieri rule (Theorem 52) to obtain

$$h_\ell F_\lambda = \Omega\big(g_{(1^\ell)} g_{\lambda^{\omega_k}}^{(k)}\big) = \sum_{(\mu,\rho)\in\mathcal{E}^k_{\lambda^{\omega_k},\ell}} (-1)^{|\mu|-|\lambda|-\ell}\Omega g_\mu^{(k)} = \sum_{(\mu,\rho)\in\mathcal{E}^k_{\lambda^{\omega_k},\ell}} (-1)^{|\mu|-|\lambda|-\ell} F_{\mu^{\omega_k}} \tag{83}$$

$$= \sum_{(\mu^{\omega_k},\rho)\in\mathcal{E}^k_{\lambda^{\omega_k},\ell}} (-1)^{|\mu|-|\lambda|-\ell} F_\mu = \sum_{(\mu,\rho')\in\mathcal{H}^k_{\lambda,\ell}} (-1)^{|\mu|-|\lambda|-\ell} F_\mu. \tag{84}$$

By Theorem 26, the iteration of this expression from $F_0 = \Omega g_0 = 1$ matches the iteration of the row Pieri rule (55) from $g_0 = 1$. Thus, F_μ satisfies

$$h_\lambda = \sum_{\substack{\mu\in\mathcal{P}^k\\ |\mu|\leqslant|\lambda|}} \mathcal{K}^{(k)}_{\mu\lambda} F_\mu \tag{85}$$

implying that $F_\mu = g_\mu^{(k)}$ by Definition 38 of the k-K-Schur functions. $\square$

The result can also be translated into the language of affine permutations since $\mathfrak{c}(\lambda)' = \mathfrak{c}(\lambda^{\omega_k})$.

Corollary 62. *For any $w \in \tilde{S}^0_{k+1}$,*

$$\Omega g_w^{(k)} = g_{w'}^{(k)}$$

where w' is obtained by replacing s_i in w with s_{k+1-i}.

9. Computability

The notion of affine s-v strips and Theorem 26 give an efficient recursive algorithm to compute k-K-Schur functions and affine Grothendieck polynomials. This enabled us to check all conjectures extensively.

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